

# Non-Symmetric Jack Polynomials and Integral Kernels

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We investigate some properties of non-symmetric Jack, Hermite and Laguerre polynomials which occur as the polynomial part of the eigenfunctions for certain Calogero-Sutherland models with exchange terms. For the non-symmetric Jack polynomials, the constant term normalization  $\mathcal{N}_\eta$  is evaluated using recurrence relations, and  $\mathcal{N}_\eta$  is related to the norm for the non-symmetric analogue of the power-sum inner product. Our results for the non-symmetric Hermite and Laguerre polynomials allow the explicit determination of the integral kernels which occur in Dunkl's theory of integral transforms based on reflection groups of type  $A$  and  $B$ , and enable many analogues of properties of the classical Fourier, Laplace and Hankel transforms to be derived. The kernels are given as generalized hypergeometric functions based on non-symmetric Jack polynomials. Central to our calculations is the construction of operators  $\hat{\Phi}$  and  $\hat{\Psi}$ , which act as lowering-type operators for the non-symmetric Jack polynomials of argument  $x$  and  $x^2$  respectively, and are the counterpart to the raising-type operator  $\Phi$  introduced recently by Knop and Sahi.

## 1 Introduction

Non-symmetric Jack polynomials occur as the polynomial part of the eigenfunctions of the Calogero-Sutherland model on a circle with exchange terms. This means, in particular, that they are eigenfunctions of the transformed Hamiltonian

$$H^{(C)} = \sum_{j=1}^n x_j^2 \frac{\partial^2}{\partial x_j^2} + \frac{2}{\alpha} \sum_{1 \leq j < k \leq n} \frac{x_j x_k}{x_j - x_k} \left[ \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) - \frac{1 - s_{jk}}{x_j - x_k} \right] \quad (1.1)$$

Here  $s_{jk}$  is the operator which acts on functions by exchanging the  $j$ 'th and  $k$ 'th coordinates. The non-symmetric Jack polynomials  $E_\eta(x)$ ,  $x := (x_1, \dots, x_n)$  were introduced by Opdam [28] and their properties have been expounded upon in [18, 29]. (Their  $q$ -analogues, the non-symmetric Macdonald polynomials were introduced in [23] and have also received attention in the literature [4, 25]). In what follows, we shall mainly be following the notation of Knop and Sahi [18, 29].

The  $E_\eta(x)$  are labelled by an  $n$ -tuple  $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{N}^n$  and are uniquely defined as being the simultaneous eigenfunctions of the (mutually commuting) Cherednik operators  $\xi_i$  defined by

$$\xi_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{p < i} \frac{x_i}{x_i - x_p} (1 - s_{ip}) + \sum_{p > i} \frac{x_p}{x_i - x_p} (1 - s_{ip}) + 1 - i \quad (1.2)$$

and by the fact that they have an expansion of the form

$$E_\eta(x) = x^\eta + \sum_{\nu < \eta} a_{\eta\nu} x^\nu. \quad (1.3)$$

Here the partial order  $<$  on  $n$ -tuples is defined for  $\eta \neq \nu$  by

$$\nu < \eta \quad \text{iff} \quad \nu^+ < \eta^+ \quad \text{or in the case } \nu^+ = \eta^+ \quad \nu < \eta$$

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where  $\eta^+$  is the unique partition associated with  $\eta$  obtained from permuting its entries, and  $<$  is the usual dominance order for  $n$ -tuples i.e.  $\nu < \eta$  iff  $\sum_{i=1}^p (\eta_i - \nu_i) \geq 0$ , for all  $1 \leq p \leq n$ . Indeed, we have the eigenvalue equation  $\xi_i E_\eta = \bar{\eta}_i E_\eta$ , where [29]

$$\bar{\eta}_i = \alpha \eta_i - \#\{k < i \mid \eta_k \geq \eta_i\} - \#\{k > i \mid \eta_k > \eta_i\} \quad (1.4)$$

The Cherednik operators are self-adjoint with respect to the inner product [3]

$$\langle f, g \rangle_C = \text{C.T.} \left( f(x) g(x^{-1}) \prod_{i \neq j} \left( 1 - \frac{x_i}{x_j} \right)^{1/\alpha} \right) \quad (1.5)$$

with C.T. meaning “the constant term of” in the Laurent polynomial expansion for  $1/\alpha \in \mathbb{N}$ , and C.T. defined as a Fourier integral with  $x_j = e^{2\pi i \theta_j}$ ,  $0 \leq \theta_j \leq 1$ , for general  $1/\alpha \geq 0$ . This self-adjointness, along with the form of the eigenvalues given in (1.4) implies that the non-symmetric Jack polynomials are orthogonal with respect to the above inner product. The value of  $\langle E_\eta, E_\eta \rangle_C$  has been computed by Opdam in [28] (and in [23, 4] for the  $q$ -case). One of the new results of this work, given in Section 2, is a different evaluation of  $\langle E_\eta, E_\eta \rangle_C$ , utilizing the operator  $\Phi$  introduced in [18].

Recently we began a study of the eigenfunctions of the Calogero-Sutherland model in an external harmonic potential with exchange terms, associated with the roots systems of type  $A$  and  $B$  [2]. The transformed Hamiltonians take the form

$$H^{(H)} := \sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} - 2x_j \frac{\partial}{\partial x_j} \right) + \frac{2}{\alpha} \sum_{j < k} \frac{1}{x_j - x_k} \left[ \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) - \frac{1 - s_{jk}}{x_j - x_k} \right] \quad (1.6)$$

$$H^{(L)} := \sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} + \left( \frac{2a+1}{x_j} - 2x_j \right) \frac{\partial}{\partial x_j} \right) + \frac{4}{\alpha} \sum_{j < k} \frac{1}{x_j^2 - x_k^2} \left[ \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) - \frac{x_j^2 + x_k^2}{x_j^2 - x_k^2} (1 - s_{jk}) \right], \quad (1.7)$$

where to obtain (1.7) we have set  $y_j = x_j^2$  in [2, (1.17)] and multiplied through by 4, and their eigenfunctions are called non-symmetric Hermite and Laguerre polynomials, denoted  $E_\eta^{(H)}(x)$  and  $E_\eta^{(L)}(x^2)$  respectively. In ref. [2] we constructed a set of commuting operators for  $H^{(H)}$ , which have  $\{E_\eta^{(H)}\}$  as simultaneous eigenfunctions, and which are self adjoint with respect to the inner product

$$\langle f, g \rangle_H := \prod_{l=1}^n \int_{-\infty}^{\infty} dx_l e^{-x_l^2} \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2/\alpha} f g \quad (1.8)$$

A consequence of this construction is the orthogonality relation  $\langle E_\eta^{(H)}, E_\nu^{(H)} \rangle_H = \mathcal{N}_\eta^{(H)} \delta_{\eta, \nu}$ , where  $\mathcal{N}_\eta^{(H)}$  is the norm. Similarly, we constructed a set of commuting operators for  $H^{(L)}$ , which are self adjoint with respect to

$$\langle f, g \rangle_L := 2^n \prod_{l=1}^n \int_{-\infty}^{\infty} dx_l e^{-x_l^2} |x_l|^{2a+1} \prod_{1 \leq j < k \leq n} |x_k^2 - x_j^2|^{2/\alpha} f(x^2) g(x^2) \quad (1.9)$$

In this paper, we continue our study of the non-symmetric Jack, Hermite and Laguerre polynomials. In Section 2 we review some results concerning the non-symmetric Jack polynomials, in particular those due to Knop and Sahi [18, 29], which are relevant to the calculations in later sections. Using these results, we provide a new proof of the evaluation of the norm  $\langle E_\eta, E_\eta \rangle_C$ . We also give a non-symmetric analogue of a generalization due to Kadell [16] of the Morris constant term identity [26]. In the course of deriving this result we obtain a formula relating the norm  $\langle E_\eta, E_\eta \rangle_C$  and the norm  $\langle E_\eta, E_\eta \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the non-symmetric analogue of the power-sum inner product specified in [6, 29].

In Section 3, we turn our attention to the non-symmetric Hermite polynomials  $E_\eta^{(H)}$ , beginning with a brief review of the pertinent results in [2]. We then proceed to construct certain operators which are sufficient to generate all  $E_\eta^{(H)}$  by recurrence. This enables us to compute the norm  $\mathcal{N}_\eta^{(H)}$  for non-symmetric Hermite polynomials. Our attention is then directed towards a construction of Dunkl's [11, 12] non-symmetric kernel  $\mathcal{K}_A(x; y)$  (an analogue of the (symmetric) generalized hypergeometric series

${}_0\mathcal{F}_0(x; y)$  [33]), which is used to define generalizations of the Fourier and Laplace transforms. This kernel also allows us to derive an exponential formula, a generating function, and integral formulae for the non-symmetric Hermite polynomials in complete analogy with the symmetric case [1]. Indeed, we show that  ${}_0\mathcal{F}_0(x; y)$  can be constructed from  $\mathcal{K}_A(x; y)$  by symmetrization. The analysis of Section 3 is repeated, albeit more succinctly, in Section 4 for the Laguerre case.

## 2 The Jack case

We begin by reviewing some of the results in [18, 29]. A fundamental result concerns the action of the elementary transpositions  $s_i := s_{i,i+1}$  on the non-symmetric Jack polynomials, which is given by

**Lemma 2.1** *Let  $\bar{\eta}_i$  be the eigenvalue of  $\xi_i$  on the non-symmetric Jack polynomial  $E_\eta$ , and let  $\delta_{i,\eta} := \bar{\eta}_i - \bar{\eta}_{i+1}$ . Then the action of  $s_i$  is given by*

$$s_i E_\eta = \begin{cases} \frac{1}{\delta_{i,\eta}} E_\eta + \left(1 - \frac{1}{\delta_{i,\eta}^2}\right) E_{s_i \eta} & \eta_i > \eta_{i+1} \\ E_\eta & \eta_i = \eta_{i+1} \\ \frac{1}{\delta_{i,\eta}} E_\eta + E_{s_i \eta} & \eta_i < \eta_{i+1} \end{cases}$$

This is a consequence of the following relations between the Cherednik operators and the transpositions  $s_i$

$$\xi_i s_i - s_i \xi_{i+1} = 1, \quad \xi_{i+1} s_i - s_i \xi_i = -1, \quad [\xi_i, s_j] = 0, \quad j \neq i, i+1 \quad (2.1)$$

Knop and Sahi also introduced a remarkable operator  $\Phi$ , defined by

$$\Phi = x_n s_{n-1} \cdots s_2 s_1 = s_{n-1} \cdots s_i x_i s_{i-1} \cdots s_1 \quad (2.2)$$

which enjoys the following properties

**Lemma 2.2**

$$\begin{aligned} \xi_j \Phi &= \Phi \xi_{j+1} & 1 \leq j \leq n-1 \\ \xi_n \Phi &= \Phi (\xi_1 + \alpha) \\ \Phi E_\eta(x) &= E_{\Phi \eta}(x) \end{aligned}$$

where  $\Phi \eta := (\eta_2, \eta_3, \dots, \eta_n, \eta_1 + 1)$

As noted in [18], these results imply that the operators  $s_i$ ,  $1 \leq i \leq n$  and  $\Phi$  are sufficient to generate all  $E_\eta$ . As an application, Sahi [29] was subsequently able to evaluate  $E_\eta$  at the point  $x_1 = x_2 = \cdots = x_n = 1$ . To write down this result, which is required below, we need some additional notation. For a node  $s = (i, j)$  in an  $n$ -tuple  $\eta$ , define the arm length  $a(s)$ , arm colength  $a'(s)$ , leg length  $l(s)$  and leg colength  $l'(s)$  by

$$\begin{aligned} a(s) &= \eta_i - j & l(s) &= \#\{k > i | j \leq \eta_k \leq \eta_i\} + \#\{k < i | j \leq \eta_k + 1 \leq \eta_i\} \\ a'(s) &= j - 1 & l'(s) &= \#\{k > i | \eta_k > \eta_i\} + \#\{k < i | \eta_k \geq \eta_i\} \end{aligned} \quad (2.3)$$

Using these, define constants  $d_\eta := \prod_{s \in \eta} d(s)$ ,  $d'_\eta := \prod_{s \in \eta} d'(s)$  and  $e_\eta := \prod_{s \in \eta} e(s)$  where

$$d'(s) := \alpha(a(s) + 1) + l(s) \quad e(s) := \alpha(a'(s) + 1) + n - l'(s)$$

and  $d(s) := d'(s) + 1$ . These have the following important properties

**Lemma 2.3** *We have*

$$\begin{aligned} \frac{d_{\Phi \eta}}{d_\eta} &= \frac{e_{\Phi \eta}}{e_\eta} = \bar{\eta}_1 + \alpha + n & \frac{d'_{\Phi \eta}}{d'_\eta} &= \bar{\eta}_1 + \alpha + n - 1 & \text{for all } \eta \\ e_{s_i \eta} &= e_\eta & \frac{d_{s_i \eta}}{d_\eta} &= \frac{\delta_{i,\eta} + 1}{\delta_{i,\eta}} & \frac{d'_{s_i \eta}}{d'_\eta} &= \frac{\delta_{i,\eta}}{\delta_{i,\eta} - 1} & \text{for } \eta_i > \eta_{i+1} \end{aligned}$$

A similar relation follows in the case  $\eta_i < \eta_{i+1}$  after noting that  $\delta_{i,s_i \eta} = -\delta_{i,\eta}$ .

Using Lemmas 2.1, 2.2 and 2.3, Sahi showed that

$$E_\eta(1^n) = e_\eta / d_\eta \quad (2.4)$$

by showing that both sides of this equation satisfy the same recursions via the operators  $s_i$  and  $\Phi$  (see [4] for another proof of this result).

## 2.1 Calculation of $\langle E_\eta, E_\eta \rangle_C$

Let us now show that a similar idea works for the calculation of the norm of the non-symmetric Jack polynomials with respect to the inner product (1.8).

**Proposition 2.4** *In the case where  $k = 1/\alpha \in \mathbb{Z}^+$*

$$\langle E_\eta, E_\eta \rangle_C = \prod_{1 \leq i < j \leq n} \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_j - \bar{\eta}_i) + p}{k(\bar{\eta}_j - \bar{\eta}_i) - p - 1} \right)^{\epsilon(\bar{\eta}_j - \bar{\eta}_i)} \quad (2.5)$$

where  $\epsilon(x) = 1$  for  $x > 0$ , and  $\epsilon(x) = -1$  for  $x \leq 0$

*Proof.* First, note that the transpositions  $s_i$  are hermitian w.r.t. the inner product defined by (1.5). Thus from the definition (2.2)  $\langle \Phi f, \Phi g \rangle_C = \langle f, g \rangle_C$ , and so  $\Phi$  is an isometry. Thus from Lemma 2.2 we have

$$\langle E_{\Phi\eta}, E_{\Phi\eta} \rangle_C = \langle \Phi E_\eta, \Phi E_\eta \rangle_C = \langle E_\eta, E_\eta \rangle_C \quad (2.6)$$

Also, for the case  $\eta_i < \eta_{i+1}$ , from Lemma 2.1 we have  $E_{s_i\eta} = s_i E_\eta - \delta_{i,\eta}^{-1} E_\eta$  so that

$$\begin{aligned} \langle E_{s_i\eta}, E_{s_i\eta} \rangle_C &= \langle (s_i - \delta_{i,\eta}^{-1}) E_\eta, (s_i - \delta_{i,\eta}^{-1}) E_\eta \rangle_C \\ &= (1 + \delta_{i,\eta}^{-2}) \langle E_\eta, E_\eta \rangle_C - 2\delta_{i,\eta}^{-1} \langle E_\eta, s_i E_\eta \rangle_C \\ &= (1 - \delta_{i,\eta}^{-2}) \langle E_\eta, E_\eta \rangle_C \end{aligned} \quad (2.7)$$

where in obtaining the last line we have used Lemma 2.1 in the second term of the previous line, and used the fact that for  $\eta_i \neq \eta_{i+1}$ ,  $E_\eta$  and  $E_{s_i\eta}$  are orthogonal. Equation (2.7) immediately implies an equivalent result in the case  $\eta_i > \eta_{i+1}$ , namely

$$\langle E_{s_i\eta}, E_{s_i\eta} \rangle_C = (1 - \delta_{i,\eta}^{-2})^{-1} \langle E_\eta, E_\eta \rangle_C$$

through the obvious change of variables  $\eta \rightarrow s_i\eta$  (recall  $\delta_{i,s_i\eta} = -\delta_{i,\eta}$ ). It thus remains to show that the right hand side of (2.5),  $\text{RHS}_\eta$  say, obeys the same recursion relations (2.6) and (2.7) and that both sides have the same evaluation in the trivial case  $\eta = 0$ . For the latter property note that then  $\bar{\eta}_i = 1 - i$  and so (2.5) reduces to

$$\langle 1, 1 \rangle_C = \prod_{1 \leq i < j \leq n} \prod_{p=0}^{k-1} \left( \frac{k(i - j) - p - 1}{k(i - j) + p} \right)$$

which is a well-known constant term identity (see for example [24]).

Turning to (2.6), first note from the definition (1.4) that if  $\zeta = \Phi\eta$ , then

$$(\bar{\zeta}_1, \bar{\zeta}_2, \dots, \bar{\zeta}_n) = (\bar{\eta}_2, \bar{\eta}_3, \dots, \bar{\eta}_n, \bar{\eta}_1 + \alpha) \quad (2.8)$$

Thus we have

$$\begin{aligned} \text{RHS}_\eta &= \prod_{1 \leq i < j \leq n} \prod_{p=0}^{k-1} \left( \frac{k(\bar{\zeta}_j - \bar{\zeta}_i) + p}{k(\bar{\zeta}_j - \bar{\zeta}_i) - p - 1} \right)^{\epsilon(\bar{\zeta}_j - \bar{\zeta}_i)} = \\ &\quad \prod_{2 \leq i < j \leq n} \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_j - \bar{\eta}_i) + p}{k(\bar{\eta}_j - \bar{\eta}_i) - p - 1} \right)^{\epsilon(\bar{\eta}_j - \bar{\eta}_i)} \prod_{i=2}^n \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_1 + \alpha - \bar{\eta}_i) + p}{k(\bar{\eta}_1 + \alpha - \bar{\eta}_i) - p - 1} \right)^{\epsilon(\bar{\eta}_1 + \alpha - \bar{\eta}_i)} \end{aligned} \quad (2.9)$$

If we can show that  $\epsilon(\bar{\eta}_1 + \alpha - \bar{\eta}_i) = -\epsilon(\bar{\eta}_i - \bar{\eta}_1)$  for all  $2 \leq i \leq n$ , then the second double product on the right hand side of (2.9) can be rewritten as

$$\prod_{i=2}^n \prod_{p=0}^{k-1} \left( \frac{-[k(\bar{\eta}_i - \bar{\eta}_1) - p - 1]}{-[k(\bar{\eta}_i - \bar{\eta}_1) + p]} \right)^{-\epsilon(\bar{\eta}_i - \bar{\eta}_1)} = \prod_{i=2}^n \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_i - \bar{\eta}_1) + p}{k(\bar{\eta}_i - \bar{\eta}_1) - p - 1} \right)^{\epsilon(\bar{\eta}_i - \bar{\eta}_1)}$$

which when reinserted back into (2.9) yields the required equality.

To show that  $\epsilon(\bar{\eta}_1 + \alpha - \bar{\eta}_i) = -\epsilon(\bar{\eta}_i - \bar{\eta}_1)$ , we consider two cases. In the first case, suppose  $\epsilon(\bar{\eta}_1 + \alpha - \bar{\eta}_i) = -1$ , i.e.  $\bar{\eta}_1 + \alpha \leq \bar{\eta}_i$ . Then  $\bar{\eta}_i - \bar{\eta}_1 \geq \alpha > 0$ , so that  $\epsilon(\bar{\eta}_i - \bar{\eta}_1) = 1$  as required. In the second case, where

$\epsilon(\bar{\eta}_1 + \alpha - \bar{\eta}_i) = 1$ , we have  $\bar{\eta}_1 + \alpha > \bar{\eta}_i$ . Note that we can always write  $\bar{\eta}_1 - \bar{\eta}_i = a\alpha + b$  for some integers  $a$  and  $b$ . Then using the fact that  $\alpha = 1/k$ , we have  $\bar{\eta}_1 + \alpha > \bar{\eta}_i \Leftrightarrow bk > -a - 1$ . But  $b$ ,  $k$  and  $a$  are all integers, so that this later inequality is equivalent to  $bk \geq -a$  which in turn is equivalent to  $\bar{\eta}_i - \bar{\eta}_1 \leq 0$ , and hence  $\epsilon(\bar{\eta}_i - \bar{\eta}_1) = -1$ .

Turning to (2.7), note that if  $\nu = s_i\eta$ , then

$$(\bar{\nu}_1, \dots, \bar{\nu}_{i-1}, \bar{\nu}_i, \bar{\nu}_{i+1}, \bar{\nu}_{i+2}, \dots, \bar{\nu}_n) = (\bar{\eta}_1, \dots, \bar{\eta}_{i-1}, \bar{\eta}_{i+1}, \bar{\eta}_i, \bar{\eta}_{i+2}, \dots, \bar{\eta}_n)$$

Moreover, when  $\eta_i < \eta_{i+1}$ , then  $\bar{\eta}_{i+1} - \bar{\eta}_i \geq \alpha(\eta_{i+1} - \eta_i) + 1 > 0$  with a similar result occurring when  $\eta_i > \eta_{i+1}$ , so that in all cases  $\bar{\eta}_{i+1} - \bar{\eta}_i \neq 0$ . The upshot of all of this is that we always have  $\epsilon(\bar{\eta}_i - \bar{\eta}_{i+1}) = -\epsilon(\bar{\eta}_{i+1} - \bar{\eta}_i)$ . From these considerations we see that

$$\begin{aligned} \prod_{p=0}^{k-1} \left( \frac{k(\bar{\nu}_{i+1} - \bar{\nu}_i) + p}{k(\bar{\nu}_{i+1} - \bar{\nu}_i) - p - 1} \right)^{\epsilon(\bar{\nu}_{i+1} - \bar{\nu}_i)} &= \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_{i+1} - \bar{\eta}_i) + p + 1}{k(\bar{\eta}_{i+1} - \bar{\eta}_i) - p} \right)^{\epsilon(\bar{\eta}_{i+1} - \bar{\eta}_i)} \\ &= (1 - \delta_{i,\eta}^{-2})^{\epsilon(\bar{\eta}_{i+1} - \bar{\eta}_i)} \prod_{p=0}^{k-1} \left( \frac{k(\bar{\eta}_{i+1} - \bar{\eta}_i) + p}{k(\bar{\eta}_{i+1} - \bar{\eta}_i) - p - 1} \right)^{\epsilon(\bar{\eta}_{i+1} - \bar{\eta}_i)} \end{aligned}$$

which in turn implies that the right hand side of (2.5) satisfies the relation (2.7).  $\square$

## 2.2 Generalization of the Macdonald-Kadell-Kaneko integral

The Macdonald-Kadell-Kaneko integral, first conjectured by Macdonald [22] and subsequently proved by Kadell [16] and Kaneko [17], relates the Selberg integral [30] to the Jack polynomial. Here we will derive the analogue of this result, relating the Selberg integral to the non-symmetric Jack polynomial.

Our derivation is based on the generalized binomial theorem [32, 16, 17]

$$\prod_{i=1}^n \frac{1}{(1 - x_i)^r} = \sum_{\kappa} \frac{\alpha^{|\kappa|} [r]_{\kappa}^{(\alpha)}}{j_{\kappa}} J_{\kappa}^{(\alpha)}(x) \quad (2.10)$$

where

$$[r]_{\kappa}^{(\alpha)} := \prod_{j=1}^n \frac{\Gamma(r - (j-1)/\alpha + \kappa_j)}{\Gamma(r - (j-1)/\alpha)} \quad (2.11)$$

Let us introduce the non-symmetric Jack polynomials with a different normalization namely  $F_{\eta} := d_{\eta} E_{\eta}$ , and define constants  $f_{\eta} := d_{\eta} d'_{\eta}$ . From [29, 23] we know that

$$\frac{1}{j_{\kappa}} J_{\kappa}^{(\alpha)}(x) = \sum_{\eta: \eta^+ = \kappa} \frac{1}{f_{\eta}} F_{\eta}(x) \quad (2.12)$$

Hence from (2.10) we have

$$\prod_{i=1}^n \frac{1}{(1 - x_i)^r} = \sum_{\eta} \frac{\alpha^{|\eta|} [r]_{\eta^+}^{(\alpha)}}{f_{\eta}} F_{\eta}(x) \quad (2.13)$$

Recall that  $\{F_{\eta}\}$  is an orthogonal set of functions with respect to the constant-term inner product (1.5), and that it constitutes a basis for analytic functions. Hence letting  $\Delta(x) := \prod_{j \neq k} (1 - \frac{x_j}{x_k})^{2/\alpha}$  we can write

$$\prod_{i=1}^n \frac{1}{(1 - x_i^{-1})^r} = \sum_{\eta} \frac{\text{C.T.}(\prod_{i=1}^n (1 - x_i^{-1})^{-r} F_{\eta}(x) \Delta(x))}{\text{C.T.}(F_{\eta}(x^{-1}) F_{\eta}(x) \Delta(x))} F_{\eta}(x^{-1}) \quad (2.14)$$

Comparing (2.13) and (2.14) gives, after replacing  $1/x$  by  $x$  in (2.14),

$$\text{C.T.}(\prod_{i=1}^n (1 - x_i^{-1})^{-r} F_{\eta}(x) \Delta(x)) = \frac{\alpha^{|\eta|} [r]_{\eta^+}^{(\alpha)}}{f_{\eta}} \text{C.T.}(F_{\eta}(x^{-1}) F_{\eta}(x) \Delta(x)) \quad (2.15)$$

Our next task is to manipulate the left hand side of (2.15) so that  $(1 - x_i^{-1})^{-r}$  is replaced by  $(1 - x_i)^a (1 - x_i^{-1})^b$ . We require

**Lemma 2.5** *We have*

$$x^p E_\eta(x) = E_{\eta+p}(x)$$

where  $\eta + p := (\eta_1 + p, \eta_2 + p, \dots, \eta_n + p)$ , and  $x^p := (x_1 x_2 \cdots x_n)^p$ .

*Proof.* Using the Cherednik operators (1.2) and the corresponding eigenvalue equation for  $E_\eta(x)$ , we have

$$\xi_j (x^p E_\eta) = (\bar{\eta}_j + \alpha p) x^p E_\eta.$$

However, from the definition (1.4) we have  $\bar{\eta}_j + \alpha p = \overline{(\eta + p)}_j$  so that  $x^p E_\eta$  must be a constant multiple of  $E_{\eta+p}$ . Examination of the leading terms shows that this constant is 1 and the result then follows.  $\square$

Consider (2.15) with  $\eta$  replaced by  $\eta + a$ . Now

$$F_{\eta+a}(x) = d_{\eta+a} E_{\eta+a}(x) = d_{\eta+a} x^a E_\eta(x)$$

by Lemma 2.5. Also, set  $r = -a - b$  and note that

$$x_i^a (1 - \frac{1}{x_i})^{a+b} = (-1)^a (1 - x_i)^a (1 - \frac{1}{x_i})^b$$

This gives

$$\text{C.T.} \left( \prod_{i=1}^n (1 - x_i)^a (1 - \frac{1}{x_i})^b E_\eta(x) \Delta(x) \right) = (-\alpha)^{an} \alpha^{|\eta|} [-a - b]_{\eta+a}^{(\alpha)} \frac{d_{\eta+a}}{f_{\eta+a}} \text{C.T.} (E_\eta(x^{-1}) E_\eta(x) \Delta(x)) \quad (2.16)$$

The dependence on  $a$  in  $d_{\eta+a}/f_{\eta+a}$  can be determined by using

**Lemma 2.6** *We have*

$$E_\eta \left( \frac{1}{x} \right) = E_{-\underline{\eta}}(\underline{x})$$

where  $\underline{x} := (x_n, x_{n-1}, \dots, x_1)$ ,  $\underline{\eta} := (\eta_n, \eta_{n-1}, \dots, \eta_1)$  and  $E_{-\underline{\eta}}$  is interpreted according to Lemma 2.5.

*Proof.* Let  $y = 1/x$ . Then

$$\xi_i^{(y)} E_\eta(y) = \bar{\eta}_i E_\eta(y).$$

Since  $\frac{\partial}{\partial y_i} = -x_i^2 \frac{\partial}{\partial x_i}$ , we have

$$\begin{aligned} \xi_i^{(y)} &= -\alpha x_i \frac{\partial}{\partial x_i} - \sum_{p < i} \frac{x_p}{x_i - x_p} (1 - s_{ip}) - \sum_{p > i} \frac{x_i}{x_i - x_p} (1 - s_{ip}) + 1 - i \\ &= -\alpha \hat{D}_i^{(x)} \end{aligned}$$

where  $\hat{D}_i$  is Hikami's version [15] of the Cherednik operator (see ref. [2, eq. (2.2) and the remarks which follow]). Hence

$$\alpha \hat{D}_i^{(x)} E_\eta \left( \frac{1}{x} \right) = -\bar{\eta}_i E_\eta \left( \frac{1}{x} \right).$$

But the leading term of  $E_\eta(\frac{1}{x})$  is  $(\frac{1}{x})^\eta$  as is the leading term of  $E_{-\underline{\eta}}(\underline{x})$  which is also an eigenfunction of  $\hat{D}_i^{(x)}$ . Hence the equality.  $\square$

Now

$$\begin{aligned} \text{C.T.} \left( \prod_{i=1}^n (1 - x_i)^a (1 - \frac{1}{x_i})^b E_\eta(x) \Delta(x) \right) &= \text{C.T.} \left( \prod_{i=1}^n (1 - \frac{1}{x_i})^a (1 - x_i)^b E_\eta(\underline{x}^{-1}) \Delta(x) \right) \\ &= \text{C.T.} \left( \prod_{i=1}^n (1 - \frac{1}{x_i})^a (1 - x_i)^b E_{-\underline{\eta}}(\underline{x}) \Delta(x) \right) \end{aligned}$$

where to obtain the second line we have used the invariance of the constant-term operation under  $x_i \rightarrow 1/x_i$  and permutations of the variables, along with the symmetry of the terms appearing in the integrand

(excluding of course  $E_\eta$ ), and to get the last line we have used Lemma 2.6. Hence (2.16) is unchanged if we interchange  $a$  and  $b$  and replace  $\eta$  by  $-\underline{\eta}$ . Equating the corresponding right hand sides of (2.16) gives

$$(-\alpha)^{an} \alpha^{|\eta|} [-a-b]_{\eta^++a}^{(\alpha)} \frac{d_{\eta+a}}{f_{\eta+a}} = (-\alpha)^{bn} \alpha^{-|\eta|} [-a-b]_{-\underline{\eta}^++b}^{(\alpha)} \frac{d_{-\underline{\eta}+b}}{f_{-\underline{\eta}+b}} \quad (2.17)$$

where we have used the fact that

$$\text{C.T.} \left( E_\eta(x^{-1}) E_\eta(x) \Delta(x) \right) = \text{C.T.} \left( E_{-\underline{\eta}}(x^{-1}) E_{-\underline{\eta}}(x) \Delta(x) \right).$$

Setting  $a = 0$  in (2.17) gives

$$(-\alpha)^{bn} \alpha^{-|\eta|} \frac{d_{-\underline{\eta}+b}}{f_{-\underline{\eta}+b}} = \alpha^{|\eta|} \frac{[-b]_{\eta^+}^{(\alpha)}}{[-b]_{-\underline{\eta}^++b}^{(\alpha)}} \frac{d_\eta}{f_\eta} \quad (2.18)$$

Substituting (2.18) into (2.16) with  $a$  and  $b$  interchanged on the right hand side and  $\eta$  replaced by  $-\underline{\eta}$  gives

$$\text{C.T.} \left( \prod_{i=1}^n (1-x_i)^a (1-\frac{1}{x_i})^b E_\eta(x) \Delta(x) \right) = \alpha^{|\eta|} \frac{[-b]_{\eta^+}^{(\alpha)} [-a-b]_{-\underline{\eta}^++b}^{(\alpha)}}{[-b]_{-\underline{\eta}^++b}^{(\alpha)}} \frac{d_\eta}{f_\eta} \mathcal{N}_\eta \quad (2.19)$$

where  $\mathcal{N}_\eta := \text{C.T.} (E_\eta(x^{-1}) E_\eta(x) \Delta(x))$ . In (2.19), the dependence on  $a$  and  $b$  on the right hand side is explicitly displayed by virtue of (2.11). This allows the  $a, b \rightarrow \infty$  asymptotics to be computed, which leads to a relationship between  $E_\eta(1^n)$  and  $d_\eta \mathcal{N}_\eta / f_\eta$  and also allows a simplification of the right hand side of (2.19). It is convenient to first take the ratio of (2.19) to that obtained upon setting  $\eta = 0$  :

$$\begin{aligned} & \frac{\text{C.T.} \left( \prod_{i=1}^n (1-x_i)^a (1-\frac{1}{x_i})^b E_\eta(x_1, \dots, x_n) \Delta(x) \right)}{\text{C.T.} \left( \prod_{i=1}^n (1-x_i)^a (1-\frac{1}{x_i})^b \Delta(x) \right)} \\ &= \alpha^{|\eta|} \frac{[-b]_{\eta^+}^{(\alpha)} [-a-b]_{-\underline{\eta}^++b}^{(\alpha)}}{[-b]_{-\underline{\eta}^++b}^{(\alpha)}} \frac{d_\eta}{f_\eta} \frac{[-b]_b^{(\alpha)}}{[-a-b]_b^{(\alpha)}} \frac{\mathcal{N}_\eta}{\mathcal{N}_0} \end{aligned} \quad (2.20)$$

Now set  $a = b$  and take the limit  $a \rightarrow \infty$  in (2.20). On the left hand side the maximum contribution comes from the neighbourhood of  $x_1 = \dots = x_n = -1$ , and so the ratio is equal to  $E_\eta(-1^n)$  while on the right hand side, since

$$\begin{aligned} [-r]_\kappa^{(\alpha)} &= \prod_{j=1}^n \frac{\Gamma(-r-(j-1)/\alpha + \kappa_j)}{\Gamma(-r-(j-1)/\alpha)} \\ &= \prod_{j=1}^n \frac{\sin \pi(-r-(j-1)/\alpha) \Gamma(1+r+(j-1)/\alpha)}{\sin \pi(-r-(j-1)/\alpha + \kappa_j) \Gamma(1+r+(j-1)/\alpha - \kappa_j)} \\ &= (-1)^{|\kappa|} \prod_{j=1}^n \frac{\Gamma(1+r+(j-1)/\alpha)}{\Gamma(1+r+(j-1)/\alpha - \kappa_j)} \end{aligned}$$

and  $\Gamma(x+u)/\Gamma(x) \sim x^u$  as  $x \rightarrow \infty$ , we see that

$$\begin{aligned} [-b]_{\eta^+}^{(\alpha)} &\sim (-1)^{|\eta|} b^{|\eta|} \\ \frac{[-a-b]_{-\underline{\eta}^++b}^{(\alpha)}}{[-a-b]_b^{(\alpha)}} &= (-1)^{|\eta|} \frac{1}{[1+a+(n-1)/\alpha]_{\eta^+}^{(\alpha)}} \sim (-1)^{|\eta|} a^{-|\eta|} \\ \frac{[-b]_b^{(\alpha)}}{[-b]_{-\underline{\eta}^++b}^{(\alpha)}} &= (-1)^{|\eta|} \prod_{j=1}^n \frac{\Gamma(1+(j-1)/\alpha)}{\Gamma(1+(j-1)/\alpha + \eta_{n+1-j}^+)} \end{aligned}$$

Hence with  $a = b$ , in the limit  $a \rightarrow \infty$  (2.20) gives

$$E_\eta(-1^n) = (-1)^{|\eta|} \left( \prod_{j=1}^n \frac{\Gamma(1 + (j-1)/\alpha)}{\Gamma(1 + (j-1)/\alpha + \eta_{n+1-j}^+)} \right) \alpha^{|\eta|} \frac{d_\eta}{e_\eta} \frac{\mathcal{N}_\eta}{\mathcal{N}_0}$$

and so

$$\alpha^{|\eta|} \frac{d_\eta}{f_\eta} \frac{\mathcal{N}_\eta}{\mathcal{N}_0} = E_\eta(1^n) \left( \prod_{j=1}^n \frac{\Gamma(1 + (j-1)/\alpha)}{\Gamma(1 + (j-1)/\alpha + \eta_{n+1-j}^+)} \right)^{-1}. \quad (2.21)$$

This relates  $d_\eta \mathcal{N}_\eta / f_\eta$  to  $E_\eta(1^n)$  and also relates the constant term norm  $\mathcal{N}_\eta$  to the norm  $\langle E_\eta, E_\eta \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the non-symmetric analogue of the power-sum inner product specified in [6, 29] (for the non-symmetric Macdonald polynomials, the result analogous to (2.21) has been given recently by Mimachi and Noumi [25]).

We can therefore rewrite (2.20) as

$$\frac{\text{C.T.} \left( \prod_{i=1}^n (1 - x_i)^a (1 - \frac{1}{x_i})^b E_\eta(x_1, \dots, x_n) \Delta(x) \right)}{\text{C.T.} \left( \prod_{i=1}^n (1 - x_i)^a (1 - \frac{1}{x_i})^b \Delta(x) \right)} = E_\eta(1^n) \frac{[-b]_{\eta^+}^{(\alpha)}}{[1 + a + (n-1)/\alpha]_{\eta^+}^{(\alpha)}} \quad (2.22)$$

Note that by multiplying both sides by  $d_\eta / e_\eta$  and summing over the distinct permutations of  $\kappa = \eta^+$  using (2.12), we get back (2.22) with  $E_\eta$  replaced by  $J_\kappa^{(\alpha)}$  on both sides, which is the formula of Kadell [16].

To obtain the integration formula of Macdonald, Kadell and Kaneko, we make use of the following lemma, proved in [14]

**Lemma 2.7** *For  $\text{Re}(\epsilon)$  large enough so that the right hand side exists,*

$$\left( \frac{\pi}{\sin \pi \epsilon} \right)^n \prod_{l=1}^n \int_{-1/2}^{1/2} d\theta_l e^{2\pi i \theta_l \epsilon} f(-e^{2\pi i \theta_1}, \dots, -e^{2\pi i \theta_n}) = \prod_{l=1}^n \int_0^1 dt_l t_l^{-1+\epsilon} f(t_1, \dots, t_n)$$

where  $f$  is a Laurent polynomial.

Notice from the derivation of (2.22) that  $a + b (= -r)$  is arbitrary, as is  $b$ . From the symmetry relation with respect to interchanging  $a$  and  $b$ , it follows that  $a$  is arbitrary as well. Now

$$\Delta(x) = \left( \prod_{1 \leq j < k \leq n} \left( 1 - \frac{x_k}{x_j} \right) \left( 1 - \frac{x_j}{x_k} \right) \right)^{1/\alpha} = (-1)^{n(n-1)/2\alpha} \prod_{j=1}^n x_j^{-(n-1)/\alpha} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2/\alpha}$$

so we have that the left hand side of (2.22) can be rewritten as

$$\begin{aligned} & \frac{\text{C.T.} \left( \prod_{i=1}^n x_i^{-(n-1)/\alpha - b} (1 - x_i)^{a+b} E_\eta(x_1, \dots, x_n) \prod_{j < k} |x_j - x_k|^{2/\alpha} \right)}{\text{C.T.} \left( \prod_{i=1}^n x_i^{-(n-1)/\alpha - b} (1 - x_i)^{a+b} \prod_{j < k} |x_j - x_k|^{2/\alpha} \right)} \\ &= \frac{\prod_{l=1}^n \int_0^1 dt_l t_l^{\lambda_1} (1 - t_l)^{\lambda_2} E_\eta(t_1, \dots, t_n) \prod_{j < k} |t_j - t_k|^{2/\alpha}}{\prod_{l=1}^n \int_0^1 dt_l t_l^{\lambda_1} (1 - t_l)^{\lambda_2} \prod_{j < k} |t_j - t_k|^{2/\alpha}} \end{aligned}$$

where the last equality follows from Lemma 2.7 with  $\epsilon = -(n-1)/\alpha - b$ ,  $\lambda_1 = -(n-1)/\alpha - b + 1$ ,  $\lambda_2 = a + b$ . i.e.  $b = -(n-1)/\alpha - \lambda_1 + 1$ ,  $a = \lambda_1 + \lambda_2 + (n-1)/\alpha + 1$ . Equating the last equality to



(2.22) gives

$$\frac{\prod_{l=1}^n \int_0^1 dt_l t_l^{\lambda_1} (1-t_l)^{\lambda_2} E_\eta(t_1, \dots, t_n) \prod_{j < k} |t_j - t_k|^{2/\alpha}}{\prod_{l=1}^n \int_0^1 dt_l t_l^{\lambda_1} (1-t_l)^{\lambda_2} \prod_{j < k} |t_j - t_k|^{2/\alpha}} = E_\eta(1^n) \frac{[\lambda_1 + (n-1)/\alpha + 1]_{\eta^+}^{(\alpha)}}{[\lambda_1 + \lambda_2 + 2(n-1)/\alpha + 2]_{\eta^+}^{(\alpha)}}$$

This formula is the generalization of the Macdonald-Kadell-Kaneko integration formula, in the form given by Kaneko [17]. The formula of [17] can be reclaimed by multiplying both sides by  $d_\eta/e_\eta$  and summing over the distinct permutations of  $\kappa = \eta^+$  using (2.12).

### 3 The Hermite case

In this section we shall construct an operator  $\widehat{\Phi}$  analogous to the operator  $\Phi$  given in (2.2), which also has a very simple action on non-symmetric Jack polynomials. The properties of  $\widehat{\Phi}$  underpin many of the results in this section.

Another key ingredient is the type  $A$  Dunkl operator given by

$$T_i = \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1}{x_i - x_p} (1 - s_{ip}). \quad (3.1)$$

which satisfies the following relations

$$\begin{aligned} [T_i, x_i] &= 1 + \frac{1}{\alpha} \sum_{p \neq i} s_{ip} & T_i s_{ip} &= s_{ip} T_p \\ [T_i, x_j] &= -\frac{1}{\alpha} s_{ij}, \quad i \neq j & [T_i, s_{jp}] &= 0 \quad i \neq j, p \end{aligned} \quad (3.2)$$

Note that we can write the Cherednik operator (1.2) in the simple form

$$\xi_i = \alpha x_i T_i + 1 - n + \sum_{p > i} s_{ip} \quad (3.3)$$

From this and (3.2) we have the following lemma [2, Lemma 3.1] which will be required later on.

**Lemma 3.1**

$$\begin{aligned} [\xi_j, T_i] &= T_i s_{ij}, & i < j \\ [\xi_j, T_i] &= T_j s_{ij}, & i > j \\ [\xi_j, T_j] &= -\alpha T_j - \sum_{p < j} s_{jp} T_j - \sum_{p > j} T_j s_{jp}. \end{aligned}$$

In [2] we showed that the operators

$$h_i = \xi_i - \frac{\alpha}{2} T_i^2 \quad (3.4)$$

are eigenoperators for the non-symmetric Hermite polynomials  $E_\eta^{(H)}$  and are mutually commuting and self-adjoint with respect to the inner product (1.8). It is straightforward to show that the operators  $h_i, s_j$  generate the same algebra as the operators  $\xi_i, s_j$ , namely

$$h_i s_i - s_i h_{i+1} = 1, \quad h_{i+1} s_i - s_i h_i = -1, \quad [h_i, s_j] = 0, \quad j \neq i, i+1$$

As a consequence, one can follow the argument of [18, Prop. 4.3] and show that

$$s_i E_\eta^{(H)} = \begin{cases} \frac{1}{\delta_{i,\eta}} E_\eta^{(H)} + \left(1 - \frac{1}{\delta_{i,\eta}^2}\right) E_{s_i \eta}^{(H)} & \eta_i > \eta_{i+1} \\ E_\eta^{(H)} & \eta_i = \eta_{i+1} \\ \frac{1}{\delta_{i,\eta}} E_\eta^{(H)} + E_{s_i \eta}^{(H)} & \eta_i < \eta_{i+1} \end{cases} \quad (3.5)$$

a result we shall use later on.

### 3.1 The operator $\hat{\Phi}$

We now define the operator  $\hat{\Phi}$  as

$$\hat{\Phi} = T_1 s_1 s_2 \cdots s_{n-1} = s_1 s_2 \cdots s_{i-1} T_i s_i s_{i+1} \cdots s_{n-1} \quad (3.6)$$

This operator obeys the following important relations

**Lemma 3.2**

$$\begin{aligned} (a) \quad \xi_j \hat{\Phi} &= \hat{\Phi} \xi_{j-1} & \text{for } 2 \leq j \leq n \\ (b) \quad \xi_1 \hat{\Phi} &= \hat{\Phi} (\xi_n - \alpha) \end{aligned}$$

*Proof.* First consider (a). From (2.1) we have for  $j \geq 2$

$$\begin{aligned} \xi_j s_1 s_2 \cdots s_{n-1} &= s_1 s_2 \cdots s_{j-2} \xi_j s_{j-1} s_j \cdots s_{n-1} \\ &= s_1 s_2 \cdots s_{j-2} (s_{j-1} \xi_{j-1} - 1) s_j \cdots s_{n-1} \\ &= s_1 s_2 \cdots s_{n-1} \xi_{j-1} - s_1 s_2 \cdots s_{j-2} s_j \cdots s_{n-1} \end{aligned}$$

From this and Lemma 3.1 we thus have

$$\begin{aligned} \xi_j \hat{\Phi} &= \xi_j T_1 s_1 s_2 \cdots s_{n-1} = T_1 (\xi_j + s_{1j}) s_1 s_2 \cdots s_{n-1} \\ &= T_1 (s_1 s_2 \cdots s_{n-1} \xi_{j-1} - s_1 s_2 \cdots s_{j-2} s_j \cdots s_{n-1} + s_{1j} s_1 s_2 \cdots s_{n-1}) \end{aligned}$$

But the permutations occurring in the last two terms in the above equation are equal, both being equal to  $(1 \ 2 \ \dots \ j-1) (j \ j+1 \ \dots \ n)$  in cycle notation. The result now follows.

Turning to (b), repeated use of (2.1) yields

$$\xi_1 s_1 s_2 \cdots s_{n-1} = s_1 s_2 \cdots s_{n-1} \xi_n + \sum_{j=1}^{n-1} s_1 \cdots s_{j-1} s_{j+1} \cdots s_{n-1}$$

Thus use of Lemma 3.1 gives

$$\begin{aligned} \xi_1 \hat{\Phi} &= T_1 \left( \xi_1 - \alpha - \sum_{p>1} s_{1p} \right) s_1 s_2 \cdots s_{n-1} \\ &= T_1 \left( s_1 s_2 \cdots s_{n-1} (\xi_n - \alpha) + \sum_{j=1}^{n-1} s_1 \cdots s_{j-1} s_{j+1} \cdots s_{n-1} - \sum_{p>1} s_{1p} s_1 s_2 \cdots s_{n-1} \right) \\ &= \hat{\Phi} (\xi_n - \alpha) \end{aligned}$$

where the last equality follows since again the permutations in the line above cancel.  $\square$

**Corollary 3.3** *The action of  $\hat{\Phi}$  on non-symmetric Jack polynomials is given by*

$$\hat{\Phi} E_\eta = \frac{1}{\alpha} \frac{d'_\eta}{d'_{\hat{\Phi}\eta}} E_{\hat{\Phi}\eta}$$

where  $\hat{\Phi}\eta := (\eta_n - 1, \eta_1, \eta_2, \dots, \eta_{n-1})$ .

*Proof.* The previous lemma implies that  $\hat{\Phi} E_\eta$  is a constant multiple of  $E_{\hat{\Phi}\eta}$ . To determine this constant, note that the leading term in  $E_{\hat{\Phi}\eta}$ , and hence in  $\hat{\Phi} E_\eta$  is a multiple of  $x^{\hat{\Phi}\eta}$ . Writing  $\hat{\Phi} = s_1 \cdots s_{n-1} T_n$ , we see that the coefficient of  $x^{\hat{\Phi}\eta}$  in  $\hat{\Phi} E_\eta$  is equal to the coefficient of  $x_1^{\eta_1} \cdots x_{n-1}^{\eta_{n-1}} x_n^{\eta_n-1}$  in  $T_n E_\eta$ . Recalling that  $\xi_n = \alpha x_n T_n + 1 - n$ , we finally deduce that the coefficient of  $x^{\hat{\Phi}\eta}$  in  $\hat{\Phi} E_\eta$  is just  $(\bar{\eta}_n + n - 1)/\alpha$ . Thus

$$\hat{\Phi} E_\eta = \frac{\bar{\eta}_n + n - 1}{\alpha} E_{\hat{\Phi}\eta}$$

However, a simple change of variables in Lemma 2.3 (recall (2.8)) tells us that

$$\frac{d'_\eta}{d'_{\widehat{\Phi}_\eta}} = \bar{\eta}_n + n - 1 \quad (3.7)$$

whence the result.  $\square$

The raising (resp. lowering) operator  $\Phi$  (resp.  $\widehat{\Phi}$ ) for the non-symmetric Jack polynomials have their counterparts for the non-symmetric Hermite polynomials. In fact  $\widehat{\Phi}$  remains a lowering operator for the  $E_\eta^{(H)}$ , but  $\Phi$  no longer has a simple action in the Hermite case. We find instead that  $\widehat{\Phi}^*$  is the appropriate raising operator for the  $E_\eta^{(H)}$ 's, where  $*$  denotes the adjoint operator with respect to the Hermite inner product (1.8). To show how this comes about, we need some preliminary results from [2, 10, 11]. Following Dunkl, define

$$\Delta_A := \sum_{i=1}^n T_i^2$$

Then we have the commutation relations

$$[\xi_i, \Delta_A] = -2\alpha T_i^2, \quad [x_i, \Delta_A] = -2T_i. \quad (3.8)$$

Also, the adjoint of the Dunkl operator  $T_i$  under the Hermite inner product (1.8) is given by

$$T_i^* = 2x_i - T_i \quad (3.9)$$

Finally, using (3.8) we have the fact that

$$[\Phi, \Delta_A] = [s_{n-1} \cdots s_1 x_1, \Delta_A] = s_{n-1} \cdots s_1 [x_1, \Delta_A] = -2s_{n-1} \cdots s_1 T_1 \quad (3.10)$$

The identities (3.9) and (3.10) allow a convenient representation of the operator  $\widehat{\Phi}^*$ , namely

$$\begin{aligned} \widehat{\Phi}^* &= s_{n-1} \cdots s_1 T_1^* = s_{n-1} \cdots s_1 (2x_1 - T_1) \\ &= 2\Phi + \frac{1}{2}[\Phi, \Delta_A] \end{aligned} \quad (3.11)$$

We are now in a position to state and prove the Hermite analogues of Lemmas 2.2 and 3.2 which take the form

**Proposition 3.4** *The operators  $h_i$  satisfy*

$$\begin{aligned} (a) \quad h_n \widehat{\Phi}^* &= \widehat{\Phi}^* (h_1 + \alpha) & h_i \widehat{\Phi}^* &= \widehat{\Phi}^* h_{i+1} & 1 \leq i \leq n-1 \\ (b) \quad h_1 \widehat{\Phi} &= \widehat{\Phi} (h_n - \alpha) & h_i \widehat{\Phi} &= \widehat{\Phi} h_{i-1} & 2 \leq i \leq n \end{aligned}$$

*Proof.* We prove only (a) as the proof of (b) is straightforward. First note that for  $1 \leq i \leq n-1$  we have  $T_i^2 \Phi = \Phi T_{i+1}^2 - \gamma$  where

$$\gamma := -\frac{1}{\alpha} s_{n-1} \cdots s_1 s_{i+1,1} (T_{i+1} + T_1).$$

This fact, along with Lemma 2.2, (3.8) and (3.11) facilitates a simple calculation which gives  $\xi_i \widehat{\Phi}^* = \widehat{\Phi}^* \xi_{i+1} - \alpha\gamma$ .

Also, the fact that the Dunkl operators commute amongst themselves, and hence with  $\Delta_A$  tells us that  $T_i^2 \widehat{\Phi}^* = \widehat{\Phi}^* T_{i+1}^2 - 2\gamma$ . Combining these two results yields the second equality in (a).

In a similar manner, one can show that

$$\xi_n \widehat{\Phi}^* = \widehat{\Phi}^* (\xi_1 + \alpha) + \alpha s_{n-1} \cdots s_1 \left( 2T_1 + \frac{1}{\alpha} \sum_{p>1} (s_{1p} T_i + s_{1p} T_p) \right)$$

and

$$T_n^2 \widehat{\Phi}^* = \widehat{\Phi}^* T_1^2 + 2 s_{n-1} \cdots s_1 \left( 2T_1 + \frac{1}{\alpha} \sum_{p>1} (s_{1p} T_i + s_{1p} T_p) \right)$$

from which the first equality in (a) follows.  $\square$

**Corollary 3.5** *The operators  $\widehat{\Phi}$  and  $\widehat{\Phi}^*$  act on the non-symmetric Hermite polynomials as*

$$\widehat{\Phi} E_\eta^{(H)} = \frac{1}{\alpha} \frac{d'_\eta}{d'_{\widehat{\Phi}\eta}} E_{\widehat{\Phi}\eta}^{(H)} \quad (3.12)$$

$$\widehat{\Phi}^* E_\eta^{(H)} = 2 E_{\Phi\eta}^{(H)} \quad (3.13)$$

*Proof.* Follows from Proposition 3.4 and examination of the leading terms on both sides of the equations.  $\square$

We are now in a position to compute the norm  $\langle E_\eta^{(H)}, E_\eta^{(H)} \rangle_H$  in the spirit of the calculation done earlier in the Jack case.

**Proposition 3.6** *We have*

$$\langle E_\eta^{(H)}, E_\eta^{(H)} \rangle_H = \frac{1}{(2\alpha)^{|\eta|}} \frac{d'_\eta e_\eta}{d_\eta} \mathcal{N}_0^{(H)} \quad (3.14)$$

where

$$\mathcal{N}_0^{(H)} := \langle 1, 1 \rangle_H = 2^{-n(n-1)/2\alpha} \pi^{n/2} \prod_{j=0}^{n-1} \frac{\Gamma(1 + (j+1)/\alpha)}{\Gamma(1 + 1/\alpha)}$$

is the ground state normalization.

*Proof.* It is clear that the operators  $s_i$  and  $\widehat{\Phi}^*$  generate all  $E_\eta^{(H)}$  via (3.5) and Corollary 3.5. Indeed, as in the Jack case, (3.5) and the orthogonality of the non-symmetric Hermite polynomials show that in the case  $\eta_i < \eta_{i+1}$

$$\langle E_{s_i\eta}, E_{s_i\eta} \rangle_H = (1 - \delta_{i,\eta}^{-2}) \langle E_\eta, E_\eta \rangle_H \quad (3.15)$$

Also, a simple change of variables in (3.12) gives

$$\widehat{\Phi} E_{\Phi\eta}^{(H)} = \frac{1}{\alpha} \frac{d'_{\Phi\eta}}{d'_\eta} E_\eta^{(H)}$$

Hence taking the inner product of both sides of (3.13) with  $E_{\Phi\eta}^{(H)}$  and dividing by 2 gives

$$\begin{aligned} \langle E_{\Phi\eta}^{(H)}, E_{\Phi\eta}^{(H)} \rangle_H &= \frac{1}{2} \langle E_{\Phi\eta}^{(H)}, \widehat{\Phi}^* E_\eta^{(H)} \rangle_H = \frac{1}{2} \langle \widehat{\Phi} E_{\Phi\eta}^{(H)}, E_\eta^{(H)} \rangle_H \\ &= \frac{1}{2\alpha} \frac{d'_{\Phi\eta}}{d'_\eta} \langle E_\eta^{(H)}, E_\eta^{(H)} \rangle_H \end{aligned} \quad (3.16)$$

It thus suffices to show that the right hand side of (3.14) satisfy the recursions (3.15) and (3.16), and is valid in the case  $\eta = 0$ . But the recursions follow immediately from Lemma 2.3, while the  $\eta = 0$  case is a well-known limiting case of Selberg's integral.  $\square$

### 3.2 Integral kernel and generating function

The operator  $\widehat{\Phi}$  has another application concerning integral kernels introduced by Dunkl [11, 12]. In particular we are able to derive a generating function for the non-symmetric Hermite polynomials, which can be used to derive an integral transform which makes explicit the theory of Dunkl in the type  $A$  case. We begin with the following lemma

**Lemma 3.7** *Let  $F = \sum_\eta A_\eta E_\eta(x) E_\eta(y)$ . Then  $s_i^{(x)} F = s_i^{(y)} F$  if and only if the coefficients  $A_\eta$  satisfy*

$$A_{s_i\eta} = \begin{cases} \left(1 - \frac{1}{\delta_{i,\eta}^2}\right) A_\eta & \eta_i > \eta_{i+1} \\ \left(1 - \frac{1}{\delta_{i,\eta}^2}\right)^{-1} A_\eta & \eta_i < \eta_{i+1} \end{cases}$$

Moreover, these two conditions on  $A_\eta$  are equivalent.

*Proof.* For a given  $i$ , split the sum occurring in  $s_i^{(x)} F$  according to whether  $\eta_i > \eta_{i+1}$ ,  $\eta_i = \eta_{i+1}$  or  $\eta_i < \eta_{i+1}$ . Apply Lemma 2.1 and collect coefficients of  $E_\eta(x)$ . The resulting terms can be identified with  $s_i^{(y)} F$  if and only if the conditions of the Lemma are satisfied. The equivalence of the two stated conditions follows through the change of variable  $\eta \rightarrow s_i \eta$ .  $\square$

Let us now define the function

$$\mathcal{K}_A(x; y) = \sum_{\eta} \alpha^{|\eta|} \frac{d_{\eta}}{d'_{\eta} e_{\eta}} E_{\eta}(x) E_{\eta}(y) \quad (3.17)$$

Its fundamental properties are given by the following result

**Theorem 3.8** *The function  $\mathcal{K}_A(x; y)$  possesses the following properties*

$$\begin{aligned} (a) \quad & s_i^{(y)} \mathcal{K}_A(x; y) = s_i^{(x)} \mathcal{K}_A(x; y) \\ (b) \quad & \widehat{\Phi}^{(y)} \mathcal{K}_A(x; y) = \Phi^{(x)} \mathcal{K}_A(x; y) \\ (c) \quad & T_i^{(y)} \mathcal{K}_A(x; y) = x_i \mathcal{K}_A(x; y) \end{aligned}$$

*Proof.*

(a) From Lemma 2.3, the constants  $A_{\eta} = \alpha^{|\eta|} \frac{d_{\eta}}{d'_{\eta} e_{\eta}}$  satisfy the conditions of Lemma 3.7 and so the result follows.

(b) Using Lemma 2.2, Lemma 2.3 and Corollary 3.3 we have

$$\begin{aligned} \widehat{\Phi}^{(y)} \mathcal{K}_A(x; y) &= \sum_{\eta} \alpha^{|\eta|} \frac{d_{\eta}}{d'_{\eta} e_{\eta}} E_{\eta}(x) \frac{1}{\alpha} \frac{d'_{\eta}}{d'_{\widehat{\Phi}\eta}} E_{\widehat{\Phi}\eta}(y) \\ &= \sum_{\nu} \alpha^{|\nu|} \frac{d_{\Phi\nu}}{d'_{\nu} e_{\Phi\nu}} E_{\Phi\nu}(x) E_{\nu}(y) \\ &= \sum_{\nu} \alpha^{|\nu|} \frac{d_{\nu}}{d'_{\nu} e_{\nu}} \Phi^{(x)} E_{\nu}(x) E_{\nu}(y) \\ &= \Phi^{(x)} \mathcal{K}_A(x; y) \end{aligned}$$

(c) From (2.2) we have

$$x_i = s_i^{(x)} s_{i+1}^{(x)} \cdots s_{n-1}^{(x)} \Phi^{(x)} s_1^{(x)} s_2^{(x)} \cdots s_{i-1}^{(x)}$$

while from (3.6) we have

$$T_i^{(y)} = s_{i-1}^{(y)} s_{i-2}^{(y)} \cdots s_1^{(y)} \widehat{\Phi}^{(y)} s_{n-1}^{(y)} s_{n-2}^{(y)} \cdots s_i^{(y)}$$

Hence using (a), (b) and the fact that operators acting on different sets of variables commute, we have

$$\begin{aligned} T_i^{(y)} \mathcal{K}_A(x; y) &= s_{i-1}^{(y)} s_{i-2}^{(y)} \cdots s_1^{(y)} \widehat{\Phi}^{(y)} s_{n-1}^{(y)} s_{n-2}^{(y)} \cdots s_i^{(y)} \mathcal{K}_A(x; y) \\ &= s_i^{(x)} s_{i+1}^{(x)} \cdots s_{n-1}^{(x)} s_{i-1}^{(y)} s_{i-2}^{(y)} \cdots s_1^{(y)} \widehat{\Phi}^{(y)} \mathcal{K}_A(x; y) \\ &= s_i^{(x)} s_{i+1}^{(x)} \cdots s_{n-1}^{(x)} \Phi^{(x)} s_{i-1}^{(y)} s_{i-2}^{(y)} \cdots s_1^{(y)} \mathcal{K}_A(x; y) \\ &= s_i^{(x)} s_{i+1}^{(x)} \cdots s_{n-1}^{(x)} \Phi^{(x)} s_1^{(x)} s_2^{(x)} \cdots s_{i-1}^{(x)} \mathcal{K}_A(x; y) \\ &= x_i \mathcal{K}_A(x; y) \end{aligned}$$

$\square$

The above Theorem enables a generating function for  $E_{\eta}^{(H)}$  to be derived. First introduce the notation

$$\widetilde{E}_k := \sum_{i=1}^n x_i^k \frac{\partial}{\partial x_i} \quad p_k(x) := \sum_{i=1}^n x_i^k. \quad (3.18)$$

Note that the Hamiltonian (1.6) can be written as  $H^{(H)} = \Delta_A - 2\widetilde{E}_1$ . From Theorem 3.8 (c) we have

$$\Delta_A^{(x)} \mathcal{K}_A(2x; z) = 4 p_2(z) \mathcal{K}_A(2x; z). \quad (3.19)$$

We also have that

$$\tilde{E}_1^{(z)} E_\eta(z) = |\eta| E_\eta(z) \quad \text{and} \quad \mathcal{K}_A(2x; z) e^{-p_2(z)} = \sum_\eta \frac{(2\alpha)^{|\eta|} d_\eta}{d'_\eta e_\eta} Q_\eta(x) E_\eta(z), \quad (3.20)$$

where  $Q_\eta(x)$  is a polynomial with leading term  $E_\eta(x)$ . Following the proof of [1, Prop. 3.1], which is Lassalle's [19] derivation of the generating function for symmetric Hermite polynomials, we see by applying the operator  $\tilde{E}_1^{(z)}$  to both sides of (3.17) and using (3.20) that  $Q_\eta(x)$  is an eigenfunction of  $H^{(H)}$  with eigenvalue  $-2|\eta|$ . Since the leading term of  $Q_\eta(x)$  is  $E_\eta(x)$  it follows that  $Q_\eta(x) = E_\eta^{(H)}(x)$ , and thus the generating function for  $E_\eta^{(H)}(x)$  is given by

**Proposition 3.9** *We have*

$$\sum_\eta \frac{(2\alpha)^{|\eta|} d_\eta}{e_\eta d'_\eta} E_\eta^{(H)}(x) E_\eta(z) = \mathcal{K}_A(2x; z) e^{-p_2(z)}.$$

An immediate application of Proposition 3.9 is to provide an alternative derivation of the norm (3.14). This also requires using the orthogonality of  $\{E_\eta^{(H)}\}$  with respect to (1.8) (see the proof of [1, Prop. 3.7] for details, where the analogous calculation is presented in the symmetric case).

We note that the analytic nature of  $\mathcal{K}_A(x, y)$  is easily established.

**Proposition 3.10** *The function  $\mathcal{K}_A(x; y)$  is an entire function of all variables.*

*Proof.* Since  $\mathcal{K}_A(x; y)$  is a sum of analytic functions (polynomials), it is sufficient to show that  $|\mathcal{K}_A(x; y)|$  is bounded. Now, since the coefficients in the monomial expansion of  $E_\eta(x)$  are positive [18],

$$|E_\eta(x)| \leq E_\eta(1^n) X^\eta = \frac{e_\eta}{d_\eta} X^\eta$$

where  $X = \max(|x_1|, \dots, |x_n|)$ , and similarly for  $|E_\eta(y)|$ . Thus

$$|\mathcal{K}_A(x; y)| \leq \sum_\eta |\alpha XY|^{|\eta|} \frac{1}{d'_\eta} E_\eta(1^n) = \sum_\kappa |\alpha XY|^{|\kappa|} \frac{1}{j_\kappa} J_\kappa(1^n),$$

where to obtain the last equality we have used the formula (2.12). But in general [32]

$$\sum_\kappa \alpha^{|\kappa|} \frac{1}{j_\kappa} J_\kappa(x) = e^{p_1(x)}$$

so we obtain the bound

$$|\mathcal{K}_A(x; y)| \leq e^{nXY} \quad (3.21)$$

and the result follows.  $\square$

Next we will show that  $\mathcal{K}_A(x; y)$  is closely related to the generalized hypergeometric function

$${}_0\mathcal{F}_0^{(\alpha)}(x; y) := \sum_\kappa \frac{C_\kappa^{(\alpha)}(x) C_\kappa^{(\alpha)}(y)}{|\kappa|! C_\kappa^{(\alpha)}(1^n)}$$

where  $C_\kappa^{(\alpha)}(x) := \alpha^{|\kappa|} |\kappa|! j_\kappa^{-1} J_\kappa^{(\alpha)}(x)$  (below we will typically omit the superscript  $(\alpha)$ ).

**Proposition 3.11** *Let  $\text{Sym}$  denote the operation of (full) symmetrization of a function of  $n$  variables, so that*

$$\text{Sym } f(x_1, \dots, x_n) := \sum_P f(x_{P(1)}, \dots, x_{P(n)}),$$

where  $P$  denotes a permutation. We have

$$\text{Sym}^{(x)} \mathcal{K}_A(x; y) = n! {}_0\mathcal{F}_0^{(\alpha)}(x; y). \quad (3.22)$$

*Proof.* From [2, eq. (2.18)], we know that

$$\text{Sym}^{(x)} E_\eta(x) = a_\eta J_{\eta^+}(x). \quad (3.23)$$

for some constant  $a_\eta$ . Therefore

$$\begin{aligned} \text{Sym}^{(x)} \mathcal{K}_A(x; y) &= \sum_\eta \alpha^{|\eta|} \frac{d_\eta}{d'_\eta e_\eta} \text{Sym}^{(x)} E_\eta(x) E_\eta(y) \\ &= \sum_\eta \alpha^{|\eta|} \frac{d_\eta}{d'_\eta e_\eta} a_\eta J_{\eta^+}(x) E_\eta(y), \end{aligned} \quad (3.24)$$

But from eq. (a) in Theorem 3.8

$$\text{Sym}^{(x)} \mathcal{K}_A(x; y) = \text{Sym}^{(y)} \mathcal{K}_A(x; y)$$

and so

$$\begin{aligned} \text{Sym}^{(x)} \mathcal{K}_A(x; y) &= \frac{1}{n!} \text{Sym}^{(y)} \text{Sym}^{(x)} \mathcal{K}_A(x; y) = \frac{1}{n!} \sum_\eta \alpha^{|\eta|} \frac{d_\eta}{d'_\eta e_\eta} a_\eta^2 J_{\eta^+}(x) J_{\eta^+}(y) \\ &= \sum_\kappa c_\kappa C_\kappa(x) C_\kappa(y) / C_\kappa(1^n) \end{aligned} \quad (3.25)$$

for some  $c_\kappa$ .

To determine  $c_\kappa$ , from eq. (c) of Theorem 3.8 and the fact that  $\sum_i T_i^{(y)} = \tilde{E}_0^{(y)}$ , we note

$$\tilde{E}_0^{(y)} \mathcal{K}_A(x; y) = p_1(x) \mathcal{K}_A(x; y)$$

But  $p_1(x)$  is symmetrical in  $x$ , so after applying  $\text{Sym}^{(x)}$  to both sides and using (3.25) we find

$$\sum_\kappa c_\kappa \tilde{E}_0^{(y)} C_\kappa(x) C_\kappa(y) = \sum_\kappa c_\kappa p_1(x) C_\kappa(x) C_\kappa(y).$$

Using known formulas for  $\tilde{E}_0^{(y)} C_\kappa(y)$  and  $p_1(x) C_\kappa(x)$  [1, eqs. (2.13a), (3.4)] we see that this equation uniquely determines the  $c_\kappa$  as  $c_\kappa = c_0 / |\kappa|!$ . But from the definition of  $\text{Sym}$  we have,  $\text{Sym } 1 = n!$ , and so  $c_0 = n!$ .  $\square$

A spin-off from Proposition 3.11 is the explicit value of the constant  $a_\eta$  in (3.23).

**Proposition 3.12** *Let  $a_\eta$  be defined by (3.23). We have*

$$a_\eta = n! \frac{e_\eta}{d_\eta J_\kappa(1^n)}. \quad (3.26)$$

*Proof.* We first note from (2.12) that

$$\frac{C_\kappa(y)}{C_\kappa(1^n)} = \frac{J_\kappa(y)}{J_\kappa(1^n)} = \frac{j_\kappa}{J_\kappa(1^n)} \sum_{\eta: \eta^+ = \kappa} \frac{1}{d_{\eta'}} E_\eta(y).$$

Substituting in the right hand side of (3.22) and equating coefficients of  $J_\kappa(x) E_\eta(y)$  with the right hand side of (3.24) multiplied by  $n!$  gives the sought result.  $\square$

The hypergeometric function  ${}_0\mathcal{F}_0(x; y)$  is related to the symmetric Hermite polynomials by a generating function analogous to Proposition 3.9, and also through an integral transform of the symmetric Jack polynomials, in which  ${}_0\mathcal{F}_0(x; y)$  is the kernel [19, 1]. Likewise,  $\mathcal{K}_A(x; y)$  also occurs as the kernel in an integral transform which relates the non-symmetric Jack and Hermite polynomials.

**Proposition 3.13** *Let*

$$d\mu^{(H)}(y) := \prod_{j=1}^n e^{-y_j^2} \prod_{1 \leq j < k \leq n} |y_j - y_k|^{2/\alpha} dy_1 \dots dy_n. \quad (3.27)$$

Then we have

$$\int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) \mathcal{K}_A(2y; w) d\mu^{(H)}(y) = \mathcal{N}_0^{(H)} e^{p_2(w)+p_2(z)} \mathcal{K}_A(2z; w) \quad (3.28)$$

$$\int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) E_\eta^{(H)}(y) d\mu^{(H)}(y) = \mathcal{N}_0^{(H)} e^{p_2(z)} E_\eta(z) \quad (3.29)$$

$$\int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; -iz) E_\eta(iy) d\mu^{(H)}(y) = \mathcal{N}_0^{(H)} e^{-p_2(z)} E_\eta^{(H)}(z) \quad (3.30)$$

*Proof.* We first note from the bound (3.21) that all the integrals exist. The first formula follows by multiplying both sides by  $e^{-p_2(w)-p_2(z)}$ , using the generating function of Proposition 3.9 twice on the left hand side and then using the orthogonality of  $\{E_\eta^{(H)}(y)\}$  with respect to (1.8) to compute the integral. The resulting sum is identified as  $\mathcal{K}_A(2z; w)$ . The second formula follows from the first after multiplying by  $e^{-p_2(w)}$ , using the generating function on the left hand side and equating coefficients of  $E_\eta(w)$  on both sides, while the third follows by replacing  $z$  by  $iz$ , using the generating function on the right hand side and equating coefficients of  $E_\eta(w)$ .  $\square$

There are a number of consequences of Proposition 3.13. First we note a summation theorem, which is the non-symmetric analogue of [1, Proposition 3.9].

**Proposition 3.14** *For  $|t| < 1$  we have*

$$\sum_\eta \frac{1}{\mathcal{N}_\eta^{(H)}} E_\eta^{(H)}(w) E_\eta^{(H)}(z) t^{|\eta|} = \frac{1}{\mathcal{N}_0^{(H)}} (1-t^2)^{-nq/2} \\ \times \exp\left(-\frac{t^2}{(1-t^2)}(p_2(z) + p_2(w))\right) \mathcal{K}_A\left(\frac{2wt}{(1-t^2)^{1/2}}; \frac{z}{(1-t^2)^{1/2}}\right)$$

where  $q := 1 + (n-1)/\alpha$ .

*Proof.* This follows by substituting the integral representation (3.28) in the left hand side, as in the proof of [1, Proposition 3.9].  $\square$

The sum in Proposition 3.14 is closely related to the Green function solution of the imaginary time Schrödinger equation

$$\left(\psi_0^{(H)} H^{(H)} (\psi_0^{(H)})^{-1}\right) G = \frac{\partial}{\partial \tau} G, \quad \psi_0^{(H)} := \prod_{i=1}^n e^{-x_i^2} \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2/\alpha}, \quad (3.31)$$

which is the solution subject to the initial condition

$$G(x^{(0)}|x; \tau) \Big|_{\tau=0} = \prod_{j=1}^n \delta(x_j - x_j^{(0)}).$$

Indeed, since  $\{E_\eta^{(H)}\}$  is a complete set of eigenfunctions of  $H^{(H)}$  which are orthogonal with respect to the inner product (1.8), by the method of separation of variables we have

$$G(x^{(0)}|x; \tau) = \psi_0^{(H)}(x^{(0)}) \psi_0^{(H)}(x) \sum_\eta \frac{1}{\mathcal{N}_\eta^{(H)}} E_\eta^{(H)}(x^{(0)}) E_\eta^{(H)}(x) e^{-2\tau|\eta|}. \quad (3.32)$$

We can use (3.32) to determine the large  $x$  (or large  $y$ ) asymptotic behaviour of  $\mathcal{K}_A(x; y)$ , since for  $\tau \rightarrow 0$  the asymptotic form of  $G(x^{(0)}|x; \tau)$  must agree with the Green function solution of

$$\sum_j \frac{\partial^2}{\partial x_j^2} G = \frac{\partial}{\partial \tau} G,$$

which gives

$$G(x^{(0)}|x; \tau) \sim \left(\frac{1}{4\pi\tau}\right)^{n/2} \prod_{j=1}^n e^{-(x_j - x_j^{(0)})^2/4\tau}.$$

Substituting in (3.32) and use of Proposition 3.14 shows



**Proposition 3.15** *We have*

$$\mathcal{K}_A(x/\tau^{1/2}; y/\tau^{1/2}) \sim \frac{\pi^{-n/2} 2^{n(n-1)/2\alpha} \mathcal{N}_0^{(H)}}{(\prod_{1 \leq j < k \leq n} (x_j - x_k)(y_j - y_k)/\tau)^{1/\alpha}} \prod_{j=1}^n e^{x_j y_j / \tau}. \quad (3.33)$$

We note that (3.33) is identical to the asymptotic behaviour of  ${}_0\mathcal{F}_0(x/\tau; y/\tau)$  (with the assumption  $x_1 < \dots < x_n$  and  $y_1 < \dots < y_n$ ) [1, eq. (5.46)].

Dunkl [11, 12] has developed a theory of integral transforms within the framework of root systems. In the type A case, the kernel satisfies properties (a) and (b) of Theorem 3.8. We can see that the type A kernel of Dunkl must be precisely our  $\mathcal{K}_A(x, y)$  by further developing the consequences of Proposition 3.13. For this we require the exponential operator formula [2, 31]

$$E_\eta^{(H)} = e^{-\Delta_A/4} E_\eta \quad (3.34)$$

(an alternative derivation of this formula is afforded by using (3.19) to deduce that  $e^{-\Delta_A^{(x)}/4} \mathcal{K}_A(2x; z) = e^{-p_2(z)} \mathcal{K}_A(2x; z)$  and then using the generating function to equate coefficients of  $E_\eta^{(H)}(z)$  on both sides). Substituting (3.34) in (3.29) and (3.30), and using the fact that  $\{E_\eta^{(H)}\}$  is a basis for analytic functions, we obtain the following formulas.

**Proposition 3.16** *We have*

$$\int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) \left( e^{-\Delta_A/4} f(y) \right) d\mu^{(H)}(y) = \mathcal{N}_0^{(H)} e^{p_2(z)} f(z) \quad (3.35)$$

$$\int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) f(iy) d\mu^{(H)}(y) = \mathcal{N}_0^{(H)} e^{-p_2(z)} e^{-\Delta_A/4} f(z) \quad (3.36)$$

where  $f$  is an analytic function such that all terms converge.

Now the type A kernel of Dunkl, here denoted  $K_A(x, y)$ , has the property (3.35) with  $\mathcal{K}_A$  replaced by  $K_A$  [11, Prop. 2.1]. Since  $f$  is arbitrary we must therefore have

$$K_A(x; y) = \mathcal{K}_A(x; y). \quad (3.37)$$

Continuing the development of the implications of Proposition 3.16 we note from (3.36) that if we define an integral transform (a generalized Fourier transform) by

$$F(z) = \frac{e^{p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) f(iy) d\mu^{(H)}(y), \quad (3.38)$$

where it is assumed that the integral is absolutely convergent, then  $F$  is inverted by

$$f(z) = \exp\left(\frac{1}{4}\Delta_A\right) F(z). \quad (3.39)$$

To obtain the inversion as an integral transform, we follow the method given in ref. [1] for the symmetric case, and replace  $z$  by  $iz$  and  $f(ix)$  by  $F(x)$  in (3.36), which gives

$$f(z) = \frac{e^{-p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) F(y) d\mu^{(H)}(y) \quad (3.40)$$

Comparison with (3.39) shows

**Proposition 3.17** *Let  $F$  be given in terms of  $f$  by (3.38). Then  $f$  is given in terms of  $F$  by*

$$f(z) = \frac{1}{\mathcal{N}_0^{(H)}} e^{-p_2(z)} \int_{(-\infty, \infty)^n} \mathcal{K}_A(2y; z) F(y) d\mu^{(H)}(y) \quad (3.41)$$

There is further overlap with the theory of Dunkl. For homogeneous polynomials  $p$  and  $q$  of the same degree  $|\eta|$  say, let

$$[p, q]_H = p(T^x)q(x), \quad (3.42)$$

where  $p(T^x)$  means each variable  $x_i$  in  $p$  is replaced by the Dunkl operator  $T_i$ . If  $p$  and  $q$  have different degrees set  $[p, q]_H = 0$ . The theory of Dunkl [11, Thm. 3.10] gives that

$$[p, q]_H = \frac{2^{|\eta|}}{\mathcal{N}_0^{(H)}} \int_{(-\infty, \infty)^n} \left( e^{-\Delta_A/4} p \right) \left( e^{-\Delta_A/4} q \right) d\mu^{(H)}(x).$$

From (3.34), (3.14) and the orthogonality of  $\{E_\eta^{(H)}\}$  with respect to (1.8) we obtain

**Proposition 3.18** *We have*

$$[E_\nu, E_\eta]_H = \frac{1}{\alpha^{|\eta|}} \frac{d'_\eta e_\eta}{d_\eta} \delta_{\nu, \eta}. \quad (3.43)$$

We will conclude this subsection by presenting results, communicated to us by C. Dunkl [13], on the relationship between the pairing (3.42) and the analogue of the power sum inner product for the non-symmetric Jack polynomials. The latter inner product is defined by Sahi [29] according to  $\langle x^\eta, p_\nu \rangle = \delta_{\eta, \nu}$ , where the polynomials  $p_\eta$  are defined by the expansion

$$\prod_{i=1}^n \frac{1}{1 - x_i y_i} \prod_{i,j=1}^n \frac{1}{(1 - x_i y_j)^{1/\alpha}} = \sum_{\eta} p_\eta(x) y^\eta \quad (3.44)$$

(Sahi uses the notation  $q_\eta$  for  $p_\eta$ ; we use the latter notation for consistency with ref. [6]). Thus if  $f$  and  $g$  are homogeneous polynomials of the same degree, and

$$f(x) = \sum_{\eta} f_\eta p_\eta(x), \quad g(x) = \sum_{\eta} g_\eta p_\eta(x), \quad p_\eta(x) = \sum_{\nu} A_{\eta\nu} x^\nu, \quad (3.45)$$

then

$$\langle f, g \rangle = \sum_{\eta} \sum_{\nu} f_\eta A_{\eta\nu} g_\nu. \quad (3.46)$$

As noted in [29], the generating function (3.44) occurs in the recent work of Dunkl [6]. Also introduced in [6] is the space  $A_\lambda$  (Dunkl uses  $E_\lambda$  but this would cause confusion with the notation for the non-symmetric Jack polynomials) of homogeneous polynomials of degree  $|\lambda|$ , where each  $f \in A_\lambda$  has the additional property that

$$V \xi f = \frac{1}{[n/\alpha + 1]_\lambda^{(\alpha)}} f. \quad (3.47)$$

Here  $\xi$  is defined by  $\xi p_\eta = \frac{1}{\eta_1! \dots \eta_n!} x^\eta$ , and  $V$  is defined by the intertwining relation  $T_i V = V \frac{\partial}{\partial x_i}$  with the normalization  $V 1 = 1$  and  $[c]_\lambda^{(\alpha)}$  by (2.11). Another characterization of  $A_\lambda$  is that it is invariant under the action of  $T_i x_i$  ([6, Prop. 3.2]), and so from (3.3) and Lemma 2.1 it follows that  $A_\lambda$  is equal to  $\text{Span}\{E_\eta : \eta^+ = \lambda\}$ .

Let us now suppose  $f$  and  $g$  in (3.45) are elements of  $A_\lambda$ . Then according to (3.45), (3.47) and the defining properties of  $\xi$  and  $V$  we have

$$\begin{aligned} [f, g]_H &= \sum_{\nu'} f_{\nu'} \sum_{\nu} A_{\nu'\nu} T^\nu \left( \sum_{\eta} g_\eta p_\eta(x) \right) \\ &= [n/\alpha + 1]_\lambda^{(\alpha)} \sum_{\nu'} f_{\nu'} \sum_{\nu} A_{\nu'\nu} T^\nu \left( \sum_{\eta} V \xi g_\eta p_\eta(x) \right) \\ &= [n/\alpha + 1]_\lambda^{(\alpha)} \sum_{\nu'} f_{\nu'} \sum_{\nu} A_{\nu'\nu} \sum_{\eta} \frac{1}{\eta_1! \dots \eta_n!} g_\eta V \left( \frac{\partial}{\partial x} \right)^\nu x^\eta \\ &= [n/\alpha + 1]_\lambda^{(\alpha)} \sum_{\nu'} f_{\nu'} \sum_{\nu} A_{\nu'\nu} g_\nu \end{aligned} \quad (3.48)$$

Comparing (3.48) with (3.46) we see that

$$[f, g]_H = [n/\alpha + 1]_\lambda^{(\alpha)} \langle f, g \rangle, \quad (3.49)$$

which upon using the facts [29] that  $e_\eta = \alpha^{|\eta|} [n/\alpha + 1]_\eta$  and  $\langle E_\eta, E_\nu \rangle = (d'_\eta/d_\eta) \delta_{\nu, \eta}$  is seen to be in precise agreement with (3.43).

### 3.3 Evaluation of $e^{p_1(x)}\mathcal{K}_A(x; y)$ and generalized binomial coefficients

In the theory of the generalized hypergeometric function  ${}_0\mathcal{F}_0(x; y)$  the identity

$$e^{p_1(x)}{}_0\mathcal{F}_0(x; y) = {}_0\mathcal{F}_0(x; y+1), \quad (3.50)$$

is an immediate consequence of the identity [17, 20]

$$e^{p_1(z)}C_\kappa^{(\alpha)}(z) = \sum_\mu \binom{\mu}{\kappa} \frac{|\kappa|!}{|\mu|!} C_\mu^{(\alpha)}(z), \quad (3.51)$$

with the generalized binomial coefficients defined by

$$\frac{C_\kappa^{(\alpha)}(1+z)}{C_\kappa^{(\alpha)}(1^n)} = \sum_\sigma \binom{\kappa}{\sigma} \frac{C_\sigma^{(\alpha)}(z)}{C_\sigma^{(\alpha)}(1^n)}. \quad (3.52)$$

Analogous results hold for the function  $\mathcal{K}_A(x; y)$ , although it is the analogue of (3.50) which is derived directly.

**Proposition 3.19** *We have*

$$e^{p_1(x)}\mathcal{K}_A(x; y) = \mathcal{K}_A(x; y+1). \quad (3.53)$$

*Proof.* Consider the action of  $\xi_i^{(y+1)}$  on  $e^{p_1(x)}\mathcal{K}_A(x; y)$ . Since  $\xi_i^{(y+1)} = \xi_i^{(y)} + \alpha T_i^{(y)}$  we have

$$\begin{aligned} \xi_i^{(y+1)} e^{p_1(x)}\mathcal{K}_A(x; y) &= e^{p_1(x)} \left( \xi_i^{(y)} + \alpha T_i^{(y)} \right) \mathcal{K}_A(x; y) \\ &= e^{p_1(x)} \left( \xi_i^{(x)} + \alpha x_i \right) \mathcal{K}_A(x; y) \\ &= \xi_i^{(x)} \left( e^{p_1(x)}\mathcal{K}_A(x; y) \right) \end{aligned} \quad (3.54)$$

But from the definition of  $\mathcal{K}_A(x; y)$  we must have

$$e^{p_1(x)}\mathcal{K}_A(x; y) = \sum_\eta \alpha^{|\eta|} \frac{d_\eta}{d'_\eta e_\eta} U_\eta(y) E_\eta(x)$$

where  $U_\eta(y)$  is a polynomial with leading term  $E_\eta(y)$ . Substituting this in (3.54) and equating coefficients of  $E_\eta(x)$  we see that

$$\xi_i^{(y+1)} U_\eta(y) = \bar{\eta}_i U_\eta(y)$$

and thus  $U_\eta(y)$  is proportional to  $E_\eta(y+1)$ . The proportionality constant is unity since the leading term of  $U_\eta(y)$  is  $E_\eta(y)$ .  $\square$

We can reclaim (3.50) from (3.53) by symmetrizing both sides with respect to  $x$  using (3.22). Since (3.50) and (3.51) are equivalent, this also establishes the latter identity. A significant feature of (3.51) is that combined with the fact that the coefficients in the monomial expansion of  $C_\kappa$  are independent of the number of variables  $n$  it immediately implies the binomial coefficients  $\binom{\kappa}{\sigma}$  are independent of  $n$ . The only other published proof of the independence of the binomial coefficients on  $n$  is given in ref. [17], via a long case-by-case check, and the independence is then used in the derivation of (3.51).

In the non-symmetric case we can define generalized binomial coefficients  $\binom{\eta}{\nu}$ ,  $|\eta| \geq |\nu|$ , analogous to (3.52) by

$$\frac{E_\eta(1+z)}{E_\eta(1^n)} = \sum_\nu \binom{\eta}{\nu} \frac{E_\nu(z)}{E_\nu(1^n)} \quad (3.55)$$

Some immediate properties of these coefficients are

**Lemma 3.20**

- (i) For  $|\nu| = |\eta|$ ,  $\binom{\eta}{\nu} = 1$  for  $\nu = \eta$  and  $\binom{\eta}{\nu} = 0$  otherwise.
- (ii) Let  $(0, \dots, 0) = 0$ . Then  $\binom{\eta}{0} = 1$ .

Moreover we can provide a simple proof that

**Proposition 3.21** *The coefficients  $\binom{\eta}{\nu}$ , like their symmetric counterparts, are independent of the number of variables  $n$ .*

*Proof.* This is done in an analogous way to the method of proof of the independence of the symmetric binomial coefficients noted above. Thus we substitute the definition (3.55) in the right hand side of (3.53), and equate coefficients of  $E_\eta(y)$  on both sides to obtain the analogue of (3.51):

$$e^{p_1(x)} E_\eta(x) \alpha^{|\eta|} \frac{1}{d'_\eta} = \sum_{\nu} \alpha^{|\nu|} \frac{1}{d'_\nu} \binom{\nu}{\eta} E_\nu(x), \quad (3.56)$$

The independence of the  $\binom{\eta}{\nu}$  on  $n$  follows immediately from this identity, and the facts that the coefficients  $a_{\eta\nu}$  in (1.3) and the  $d'_\eta$  are independent of  $n$ .  $\square$

The symmetric generalized binomial coefficients can be expressed in terms of the non-symmetric ones. Indeed note that by symmetrizing (3.55) using (3.23) and (3.26), and comparing with (3.52), we obtain

$$\sum_{\nu: \nu^+ = \mu} \binom{\eta}{\nu} = \binom{\kappa}{\mu}, \quad (3.57)$$

while symmetrizing (3.56) and comparing with (3.51) gives

$$\frac{d'_\eta j_\mu J_\kappa(1^n)}{j_\kappa J_\mu(1^n)} \sum_{\nu: \nu^+ = \mu} \frac{1}{d'_\nu} \binom{\nu}{\eta} = \binom{\mu}{\kappa}. \quad (3.58)$$

For further application of the non-symmetric binomial coefficients, let  $\tilde{E}_0$  and  $\tilde{E}_2$  be defined by (3.53), and set

$$\tilde{D}_1 := D_1 - \frac{1}{\alpha} \sum_{j \neq k} \frac{x_j}{(x_j - x_k)^2} (1 - M_{jk}) \quad (3.59)$$

where

$$D_p := \sum_{j=1}^n x_j^p \frac{\partial^2}{\partial x_j^2} + \frac{2}{\alpha} \sum_{j \neq k} \frac{x_j^p}{x_j - x_k} \frac{\partial}{\partial x_j}.$$

The non-symmetric binomial coefficients can be used to compute the action of  $\tilde{E}_0$ ,  $\tilde{E}_2$  and  $\tilde{D}_1$  on the  $E_\eta$ .

**Proposition 3.22** *We have*

$$\tilde{E}_0 \frac{E_\eta(x)}{E_\eta(1^n)} = \sum_{\nu: |\nu| = |\eta| - 1} \binom{\eta}{\nu} \frac{E_\nu(x)}{E_\nu(1^n)} \quad (3.60)$$

$$\tilde{E}_2 E_\eta(x) = \frac{\alpha}{2} d'_\eta \sum_{\nu: |\nu| = |\eta| + 1} \frac{1}{d'_\nu} \binom{\nu}{\eta} (\epsilon_\nu - \epsilon_\eta - \frac{2}{\alpha} (N - 1)) E_\nu(x) \quad (3.61)$$

$$\tilde{D}_1 \frac{E_\eta(x)}{E_\eta(1^n)} = \frac{1}{2} \sum_{\nu: |\nu| = |\eta| - 1} \binom{\eta}{\nu} (\epsilon_\eta - \epsilon_\nu) \frac{E_\nu(x)}{E_\nu(1^n)} \quad (3.62)$$

where

$$\epsilon_\eta = \sum_{j=1}^n \left( \eta_j^+ (\eta_j^+ - 1) + \frac{2}{\alpha} (N - j) \eta_j^+ \right). \quad (3.63)$$

*Proof.* The action of  $\tilde{E}_0$  follows immediately from the definition (3.55) used to expand  $E_\eta(\epsilon + x)$  in the formula

$$\begin{aligned} \tilde{E}_0 E_\eta(x) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( E_\eta(\epsilon + x) - E_\eta(x) \right) \\ &= \sum_{\nu: |\nu| = |\eta| - 1} \binom{\eta}{\nu} E_\nu(x). \end{aligned} \quad (3.64)$$

To compute the action of  $\tilde{E}_2$  we follow the strategy given in ref. [27] in the symmetric case and apply the operator

$$\begin{aligned}\tilde{D}_2 &:= D_2 - \frac{1}{\alpha} \sum_{j \neq k} \frac{x_j x_k}{(x_j - x_k)^2} (1 - s_{jk}) \\ &= H^{(C)} + \frac{2}{\alpha} (N - 1) \tilde{E}_1,\end{aligned}\tag{3.65}$$

which is an eigenoperator for each  $E_\eta$  with corresponding eigenvalue  $\epsilon_\eta$  (3.63), to the identity

$$p_1(x) E_\eta(x) = \alpha d'_\eta \sum_{\nu: |\nu|=|\eta|+1} \frac{1}{d'_\nu} \binom{\nu}{\eta} E_\nu(x),\tag{3.66}$$

which follows immediately from (3.56). The action of  $\tilde{D}_1$  follows from the action of  $\tilde{E}_0$  and  $\tilde{D}_2$  and the readily verified identity  $\tilde{D}_1 = \frac{1}{2}[\tilde{E}_0, \tilde{D}_2]$ .  $\square$

The formulas of Proposition 3.22 will be used in Section 4 to derive a partial differential equation satisfied by a generalization of  $\mathcal{K}_A(x; y)$ , which has application in the derivation of generating functions for the non-symmetric Laguerre polynomials.

### 3.4 The Laplace transform

The non-symmetric generalized Laplace transform is defined by

$$\mathcal{L}[f](t) = \int_{[0, \infty)} \mathcal{K}_A(-t; x) f(x) \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2/\alpha} dx_1 \dots dx_n\tag{3.67}$$

where it is assumed the integral is absolutely convergent (note from (3.33) that for large- $x$ , and  $x$  and  $t$  suitably ordered

$$\mathcal{K}_A(-t; x) = O(e^{-t_1 x_1 - \dots - t_n x_n}).$$

In the symmetric case ( $\mathcal{K}_A(-t; x)$  replaced by  ${}_0\mathcal{F}_0(-t; x)$ ) with  $n = 2$ , this has been considered in some detail by Yan [34]. We find that the results of Yan all have non-symmetric counterparts.

A very simple example is the shift property [34, Prop. 4.10]

$$\mathcal{L}[e^{-p_1(x)} f](t) = \mathcal{L}[f](t + 1),\tag{3.68}$$

which follows from the definition and (3.53). A more fundamental result is that  $\mathcal{L}$  is injective.

**Proposition 3.23** *If  $\langle f, f \rangle_L < \infty$  and  $\mathcal{L}[f](t) = 0$  for all  $t$  then  $f = 0$  a.e..*

*Proof* This is shown by adapting the strategy of Yan. We will make use of the formula

$$P_\nu(T^t) \frac{E_\eta(t)}{E_\eta(1^n)} \Big|_{t=0} = \delta_{\nu, \eta}, \quad \text{where} \quad P_\nu(x) = \frac{(2\alpha)^{|\nu|}}{d'_\nu} E_\nu(x),$$

which follows from (3.42). Applying this formula to (3.55) gives that for all  $\eta$  with  $|\nu| \leq |\eta|$

$$P_\nu(T^t) \frac{E_\eta(t)}{E_\eta(1^n)} \Big|_{t=1} = \binom{\eta}{\nu},$$

and so

$$P_\nu(T^t) \mathcal{K}_A(-t; x) \Big|_{t=1} = (-\alpha)^{|\nu|} \frac{1}{d'_\nu} E_\nu(x) e^{-p_1(x)}$$

where the formula (3.56) has also been used. Thus

$$P_\nu(T^t) \mathcal{L}[f](t) \Big|_{t=1} = \frac{(-\alpha)^{|\nu|}}{d'_\nu} \int_{[0, \infty)} e^{-p_1(x)} E_\nu(x) f(x) \prod_{1 \leq j < k \leq n} |x_k - x_j|^{2/\alpha} dx_1 \dots dx_n.$$

Since we are assuming  $\mathcal{L}[f](t) = 0$ , this last expression must vanish, and this holds true for all  $\nu$ . But  $\{E_\nu\}$  is a basis for analytic functions, so it follows that  $f = 0$  (this can be seen by forming an appropriate linear combination of the  $E_\nu$  so as to reconstruct  $f(t)$  a.e., which gives  $\langle f, f \rangle_L \Big|_{a=0} = 0$ .)  $\square$

Further properties of the generalized Laplace transform will be discussed in the next section.

### 3.5 Relationship to Dunkl's theory of harmonic polynomials

Van Diejen [5] has shown how the symmetric generalized Hermite (and Laguerre) polynomials can be written in terms of Dunkl's generalized spherical polynomials [9]. In ref. [9] Dunkl has extended the theory of [8] to the non-symmetric case, and this allows the considerations of [5] to be similarly extended.

The  $A$  type generalized harmonic polynomials,  $\mathcal{Y}_{k,l}^A$  say, are defined by the equation

$$\Delta_A^2 \mathcal{Y}_{k,l}^A = 0 \quad (3.69)$$

and the conditions that they are homogeneous of degree  $k$  and linearly independent. The label  $l$  distinguishes linearly independent solutions for each value of  $k$ ; with  $P_k$  denoting the space of homogeneous polynomials of degree  $k$ , Dunkl [10] has shown that there are  $\dim P_k - \dim P_{k-2}$  linearly independent solutions. However, in Dunkl's theory the basis for the label  $l$  is left unspecified, and in the equations below  $l$  will be replaced by a dot.

Now, from [9, Th. 1.11] any homogeneous polynomial can be expressed in terms of certain harmonic polynomials, which themselves are specified by the homogeneous polynomial. Applying this formula to the non-symmetric Jack polynomials gives

$$E_\eta(x) = \sum_{m=0}^{[\lceil \eta \rceil / 2]} r^{2m} \mathcal{Y}_{|\eta|-2m, \cdot}^A(x), \quad (3.70)$$

where  $r := (x_1^2 + \dots + x_n^2)^{1/2}$ , with

$$\mathcal{Y}_{|\eta|-2m, \cdot}^A(x) = \frac{1}{4^m m! \binom{n/2 + n(n-1)/2\alpha + |\eta| - 2m}{m}} \tilde{T}_{|\eta|-2m}^A \left( \Delta_A^m E_\eta(x) \right), \quad (3.71)$$

$$\tilde{T}_k^A = \sum_{j=0}^{[k/2]} \frac{r^{2j}}{4^j j! \binom{-n/2 - n(n-1)/2\alpha - k + 2}{j}} \Delta_A^j. \quad (3.72)$$

Dunkl's theory also allows the non-symmetric Hermite polynomials to be expressed in terms of the harmonic polynomials. Thus from [11, Prop. 3.9] we know that

$$e^{-\Delta_A/4} |r|^{2j} \mathcal{Y}_{k, \cdot}^A(x) = (-1)^j j! L_j^{k+n(n-1)/2\alpha+n/2-1}(r^2) \mathcal{Y}_{k, \cdot}^A(x), \quad (3.73)$$

where  $L_j^a$  denotes the classical (one-variable) Laguerre polynomial, and so applying the formula (3.34) to (3.70) we obtain

$$E_\eta^{(H)}(x) = \sum_{m=0}^{[\lceil \eta \rceil / 2]} (-1)^m m! L_m^{(|\eta|-2m+n(n-1)/2\alpha+n/2-1)}(r^2) \mathcal{Y}_{|\eta|-2m, \cdot}^A(x). \quad (3.74)$$

## 4 The Laguerre Case

In this section we investigate the Laguerre case, in an analogous manner to the Hermite case of the previous section. We begin by reviewing some results in [2]. Firstly, we have the type  $B$  Dunkl operators

$$T_i^{(B)} := \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \left( \frac{1 - s_{ip}}{x_i - x_p} + \frac{1 - \sigma_i \sigma_p s_{ip}}{x_i + x_p} \right) + \frac{a + 1/2}{x_i} (1 - \sigma_i) \quad (4.1)$$

where  $\sigma_j$  is the operator which replaces  $x_j$  by  $-x_j$ . Instead of working with the operator  $T_i^{(B)}$  directly, in [2] we found it convenient to work with the operator  $B_i := \frac{1}{4} (T_i^{(B)})^2$  acting on functions of  $x^2$ , since in this case

$$B_i = x_i^2 \hat{T}_i^2 + (a + 1) \hat{T}_i + \frac{1}{\alpha} \sum_{p \neq i} s_{ip} \hat{T}_i$$

where  $\hat{T}_i$  is the type  $A$  Dunkl operator in the variables  $x^2$ :

$$\hat{T}_i = \frac{1}{2x_i} \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1 - s_{ip}}{x_i^2 - x_p^2}.$$

Moreover, the operators  $B_i$  enjoy the following property [2, Lemma 4.2]

**Lemma 4.1** *Let*

$$\hat{\xi}_j := \alpha x_j^2 \hat{T}_j + (1 - n) + \sum_{p>j} s_{jp} \quad (4.2)$$

*be the Cherednik operator (1.2) with the substitution  $x_j \rightarrow x_j^2$  ( $j = 1, \dots, N$ ). Then we have*

$$\begin{aligned} [\hat{\xi}_j, B_i] &= B_i s_{ij}, & i < j \\ [\hat{\xi}_j, B_i] &= B_j s_{ij}, & i > j \\ [\hat{\xi}_j, B_j] &= -\alpha B_j - \sum_{p<j} s_{jp} B_j - \sum_{p>j} B_j s_{jp}. \end{aligned}$$

The non-symmetric Laguerre polynomials  $E_\eta^{(L)}(x^2)$ , which are eigenfunctions of (1.7) with eigenvalue  $-4|\eta|$ , are also eigenfunctions of the operators [2]

$$l_i = \hat{\xi}_i - \alpha B_i \quad (4.3)$$

which are mutually commuting and self-adjoint with respect to the inner product (1.9). As in the Jack and Hermite cases, the operators  $l_i, s_j$  obey the relations

$$l_i s_i - s_i l_{i+1} = 1, \quad l_{i+1} s_i - s_i l_i = -1, \quad [l_i, s_j] = 0, \quad j \neq i, i+1$$

which immediately gives

$$s_i E_\eta^{(L)} = \begin{cases} \frac{1}{\delta_{i,\eta}} E_\eta^{(L)} + \left(1 - \frac{1}{\delta_{i,\eta}^2}\right) E_{s_i \eta}^{(L)} & \eta_i > \eta_{i+1} \\ E_\eta^{(L)} & \eta_i = \eta_{i+1} \\ \frac{1}{\delta_{i,\eta}} E_\eta^{(L)} + E_{s_i \eta}^{(L)} & \eta_i < \eta_{i+1} \end{cases} \quad (4.4)$$

Let  $\Psi$  be the analogue of the operator  $\Phi$  acting on functions of  $x^2$ . That is,  $\Psi := x_n^2 s_{n-1} \cdots s_2 s_1$  which clearly acts on non-symmetric Jack functions by

$$\Psi E_\eta(x^2) = E_{\Phi \eta}(x^2)$$

where, as before  $\Phi \eta = (\eta_2, \eta_3, \dots, \eta_n, \eta_1 + 1)$ .

We also introduce the non-symmetric analogue of the generalized factorial function as

$$\begin{aligned} [c]_\eta &:= \prod_{s \in \eta} (c + a'(s) - l'(s)/\alpha) \\ &= \prod_{i=1}^n (c - \eta_i + \bar{\eta}_i/\alpha)_{\eta_i^+} \end{aligned} \quad (4.5)$$

where  $a'(s), l'(s)$  are defined in (2.3). Following the arguments of [29, Lemmas 4.1, 4.2] (indeed  $e_\eta = \alpha^{|\eta|} [n/\alpha + 1]_\eta$ ) the coefficients  $[c]_\eta$  have the properties

$$[c]_{s_i \eta} = [c]_\eta \quad \text{for all } i, \quad \frac{[c]_{\Phi \eta}}{[c]_\eta} = c + \bar{\eta}_1/\alpha, \quad \frac{[c]_\eta}{[c]_{\hat{\Phi} \eta}} = c - 1 + \bar{\eta}_n/\alpha. \quad (4.6)$$

From the first property we see that

$$[c]_\eta = [c]_{\eta^+} = \prod_{i=1}^n \left(c - \frac{i-1}{\alpha}\right)_{\eta_i^+}. \quad (4.7)$$

Finally let  $\hat{\Psi} := B_1 s_1 s_2 \cdots s_{n-1}$ . Then Lemma 3.2 and Corollary 3.3 have the following analogue in the Laguerre case

**Lemma 4.2** *We have*

$$\begin{aligned}
(a) \quad \hat{\xi}_j \hat{\Psi} &= \hat{\Psi} \hat{\xi}_{j-1} & 2 \leq j \leq n \\
(b) \quad \hat{\xi}_1 \hat{\Psi} &= \hat{\Psi} (\hat{\xi}_n - \alpha) \\
(c) \quad \hat{\Psi} E_\eta(x^2) &= \frac{1}{\alpha} \frac{[a+q]_\eta}{[a+q]_{\hat{\Phi}_\eta}} \frac{d'_\eta}{d'_{\hat{\Phi}_\eta}} E_{\hat{\Phi}_\eta}(x^2)
\end{aligned}$$

where  $q := 1 + (n-1)/\alpha$ .

*Proof.* (a) and (b) follow immediately from Lemma 4.1, in analogy to the corresponding results in the previous section. To prove (c), we know from (a) and (b) that

$$\hat{\Psi} E_\eta(x^2) = c_\eta E_{\hat{\Phi}_\eta}(x^2)$$

for some constant  $c_\eta$ . An examination of the leading term shows that

$$c_\eta = (a + q - 1 + \bar{\eta}_n/\alpha) \left( \frac{\bar{\eta}_n + n - 1}{\alpha} \right)$$

This can be simplified with (3.7) and (4.6) to give the desired result.  $\square$

We now construct raising and lowering operators for the non-symmetric Laguerre polynomials in the same manner as for the Hermite case. From Dunkl [11] we have the relations

$$[x_i, \Delta_B] = -2T_i^{(B)}, \quad (T_i^{(B)})^* = 2x_i - T_i^{(B)}, \quad (4.8)$$

where  $\Delta_B := \sum_i (T_i^{(B)})^2$  and  $(T_i^{(B)})^*$  denotes the adjoint of  $T_i^{(B)}$  with respect to the Laguerre inner product (1.9), while from [2] we have

$$[\hat{\xi}_i, \Delta_B] = -4\alpha B_i$$

Using (4.8) we have the following expression for  $\hat{\Psi}^*$ :

$$\begin{aligned}
\hat{\Psi}^* = s_{n-1} \cdots s_1 B_1^* &= \frac{1}{4} s_{n-1} \cdots s_1 \left( 2x_1 - T_1^{(B)} \right)^2 \\
&= \Psi + \frac{1}{4} [\Psi, \Delta_B] + s_{n-1} \cdots s_1 B_1
\end{aligned} \quad (4.9)$$

The Laguerre analogue of Proposition 3.4 is given by

**Proposition 4.3** *The operators  $l_i$  satisfy*

$$\begin{aligned}
(a) \quad l_n \hat{\Psi}^* &= \hat{\Psi}^* (l_1 + \alpha) & l_i \hat{\Psi}^* &= \hat{\Psi}^* l_{i+1} & 1 \leq i \leq n-1 \\
(b) \quad l_1 \hat{\Psi} &= \hat{\Psi} (l_n - \alpha) & l_i \hat{\Psi} &= \hat{\Psi} l_{i-1} & 2 \leq i \leq n
\end{aligned}$$

*Proof.* We prove only the first equation appearing in (a), the others being similar. Using the representation (4.9) we have

$$\begin{aligned}
B_i \hat{\Psi}^* &= \hat{\Psi}^* B_{i+1} + 2\tau - [\Delta_B, \tau] \\
\hat{\xi}_i \hat{\Psi}^* &= \hat{\xi}_{i+1} \hat{\Psi}^* + 2\alpha\tau + s_{in} s_{n-1} \cdots s_1 B_1 + s_{n-1} \cdots s_{i+1} s_{i-1} \cdots s_1 B_{i+1}
\end{aligned}$$

where

$$\begin{aligned}
\tau &:= \frac{1}{4} (B_i \Psi - \Psi B_{i+1}) = \frac{1}{4} s_{n-1} \cdots s_1 [B_{i+1}, x_1^2] \\
&= \frac{1}{4} s_{n-1} \cdots s_1 \left( x_{i+1}^2 \hat{T}_{i+1} + x_1^2 \hat{T}_1 + a + 1 + \sum_{p \neq i+1} s_{ip} \right) \left( -\frac{1}{\alpha} s_{i+1,1} \right)
\end{aligned}$$

It remains to calculate the commutator  $[\Delta_B, \tau]$ . To this end we recall that when acting on functions of  $x^2$ , we have  $T_i^{(B)} = 2x_i \hat{T}_i$  whence from (4.8)

$$[\Delta_B, x_i^2 \hat{T}_i] = \frac{1}{2} [\Delta_B, x_i T_i^{(B)}] = 4 B_i.$$



Thus

$$[\Delta_B, \tau] = -\frac{1}{\alpha} s_{n-1} \cdots s_1 s_{i+1,1} (B_{i+1} + B_1)$$

and the result now follows using the fact that

$$s_{n-1} \cdots s_1 s_{i+1,1} = s_{n-1} \cdots s_{i+1} s_{i-1} \cdots s_1 = s_{in} s_{n-1} \cdots s_1$$

□

**Corollary 4.4** *The operators  $\widehat{\Psi}$  and  $\widehat{\Psi}^*$  act on the non-symmetric Laguerre polynomials via*

$$\begin{aligned} \widehat{\Psi} E_\eta^{(L)}(x^2) &= \frac{1}{\alpha} \frac{[a+q]_\eta}{[a+q]_{\widehat{\Phi}_\eta}} \frac{d'_\eta}{d'_{\widehat{\Phi}_\eta}} E_{\widehat{\Phi}_\eta}^{(L)}(x^2) \\ \widehat{\Psi}^* E_\eta^{(L)}(x^2) &= E_{\Phi_\eta}^{(L)}(x^2) \end{aligned}$$

We can now use the above result to calculate the norm of the functions  $E_\eta^{(L)}(x^2)$

**Proposition 4.5** *We have*

$$\left\langle E_\eta^{(L)}, E_\eta^{(L)} \right\rangle_L = \frac{[a+q]_\eta}{\alpha^{|\eta|}} \frac{d'_\eta e_\eta}{d_\eta} \mathcal{N}_0^{(L)}$$

where

$$\mathcal{N}_0^{(L)} := \langle 1, 1 \rangle_L = \alpha^{(1-N-(N-1)^2/\alpha)} \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)/\alpha) \Gamma(a+1+j/\alpha)}{\Gamma(1+1/\alpha)}.$$

*Proof.* Similar to the proof of Proposition 3.6. □

## 4.1 Generating function

Let

$$\mathcal{K}_B(x^2; y^2) = \sum_\eta \frac{\alpha^{|\eta|}}{[a+q]_\eta} \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x^2) E_\eta(y^2) \quad (4.10)$$

where  $q$  is defined as in Lemma 4.2. Then the following result is proved in the manner of Theorem 3.8

**Theorem 4.6** *The function  $\mathcal{K}_B(x^2; y^2)$  possesses the following properties*

$$\begin{aligned} (a) \quad s_i^{(y)} \mathcal{K}_B(x^2; y^2) &= s_i^{(x)} \mathcal{K}_B(x^2; y^2) \\ (b) \quad \widehat{\Psi}^{(y)} \mathcal{K}_B(x^2; y^2) &= \Psi^{(x)} \mathcal{K}_B(x^2; y^2) \\ (c) \quad B_i^{(y)} \mathcal{K}_B(x^2; y^2) &= x_i^2 \mathcal{K}_B(x^2; y^2) \end{aligned}$$

The formula (c) of Theorem 4.6 implies that

$$\Delta_B \mathcal{K}_B(x^2; y^2) = 4p_1(x^2) \mathcal{K}_B(x^2; y^2). \quad (4.11)$$

From (4.11) and the fact that  $H^{(L)} = \Delta_B - 2\tilde{E}_1$ , we can proceed as in the proof of Proposition 3.9 to derive a generating function for the non-symmetric Laguerre polynomials.

**Proposition 4.7** *We have*

$$\sum_\eta \frac{(-\alpha)^{|\eta|}}{[a+q]_\eta} \frac{d_\eta}{d'_\eta e_\eta} E_\eta^{(L)}(x) E_\eta(z) = \mathcal{K}_B(x; -z) e^{p_1(z)}. \quad (4.12)$$

Analogous to the case of the symmetric Laguerre polynomials [1, Prop. 4.3] we can use the generating function (4.12) to express the non-symmetric Laguerre polynomials as a series in non-symmetric Jack polynomials involving the generalized binomial coefficients.

**Proposition 4.8** *We have*

$$E_\eta^{(L)}(x) = \frac{(-1)^{|\eta|} [a+q]_\eta e_\eta}{d_\eta} \sum_\nu \frac{(-1)^{|\nu|}}{[a+q]_\nu} \frac{d_\nu}{e_\nu} \binom{\eta}{\nu} E_\nu(x) \quad (4.13)$$

$$E_\eta(x) = \frac{[a+q]_\eta e_\eta}{d_\eta} \sum_\nu \frac{1}{[a+q]_\nu} \frac{d_\nu}{e_\nu} \binom{\eta}{\nu} E_\nu^{(L)}(x) \quad (4.14)$$

*Proof.* The formula (4.13) follows from (4.12) by applying the identity (3.56) on the right hand side and equating coefficients of  $E_\eta(z)$ . The formula (4.14) follows from (4.12) by multiplying both sides by  $e^{-p_1(z)}$ , using the identity (3.56) with  $z$  replaced by  $-z$  on the left hand side and equating like coefficients of  $E_\eta(z)$ .

By substituting  $x = 0$  in (4.13) we obtain the explicit value of  $E_\eta^{(L)}$  at the origin.

**Corollary 4.9** *We have*

$$E_\eta^{(L)}(0) = \frac{(-1)^{|\eta|} [a+q]_\eta e_\eta}{d_\eta}.$$

## 4.2 Other generating functions

For the symmetric Laguerre polynomials we presented [1] three generating functions including the analogue of (4.12). These generating functions all rely on a partial differential equation [1, Prop. A.1] for the generalized hypergeometric function

$${}_2\mathcal{F}_1(x; y) := \sum_\kappa \frac{1}{|\kappa|!} \frac{[a]_\kappa [b]_\kappa}{[c]_\kappa} \frac{C_\kappa(x) C_\kappa(y)}{C_\kappa(1^n)}.$$

A similar partial differential equation is satisfied by the function

$${}_2\mathcal{K}_1(x; y) := \sum_\eta \alpha^{|\eta|} \frac{[a]_\eta [b]_\eta}{[c]_\eta} \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x) E_\eta(y). \quad (4.15)$$

**Proposition 4.10** *The function  ${}_2\mathcal{K}_1(x; y)$  satisfies the p.d.e.*

$$\tilde{D}_1^{(x)} F + \left(c - \frac{N-1}{\alpha}\right) \tilde{E}_0^{(x)} F - \left(a + b - \frac{N-1}{\alpha}\right) \tilde{E}_2^{(y)} F - \frac{1}{2} [\tilde{D}_2, \tilde{E}_2]^{(y)} F = ab p_1(y) F \quad (4.16)$$

where  $\tilde{E}_0, \tilde{E}_2$  are defined by (3.18),  $\tilde{D}_1$  is defined by (3.59) and  $\tilde{D}_2$  by (3.65). In fact  ${}_2\mathcal{K}_1$  is the unique solution of this p.d.e. of the form

$$F(x, y) = \sum_\eta A_{\eta+} \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x) E_\eta(y), \quad (A_0 = 1) \quad (4.17)$$

*Proof.* Substituting (4.17) in (4.16), we see that the action of all the operators on  $E_\eta(x)$  and  $E_\eta(y)$  is specified in Proposition 3.22, while  $p_1(y) E_\eta(y)$  can be written according to (3.66). Equating like coefficients of  $E_\eta(x) E_\eta(y)$ ,  $|\nu| = |\eta| + 1$  gives

$$\begin{aligned} & \left( \frac{\epsilon_\nu - \epsilon_\eta}{2} + c - \frac{N-1}{\alpha} \right) \binom{\nu}{\eta} A_{\nu+} \\ &= \left( \left( \frac{\epsilon_\nu - \epsilon_\eta}{2} - \frac{N-1}{\alpha} \right) \left( a + b - \frac{N-1}{\alpha} + \frac{\epsilon_\nu - \epsilon_\eta}{2} \right) + ab \right) \binom{\nu}{\eta} A_{\eta+} \end{aligned} \quad (4.18)$$

Let's suppose that  $\eta_1 \leq \eta_2 \leq \dots \leq \eta_N$  and  $\nu_{N-i+1} = \eta_{N-i+1} + 1$ ,  $\nu_j = \eta_j$   $j = 1, \dots, n$  ( $j \neq i$ ). Suppose furthermore that  $\eta_{N-i+1} + 1 \leq \eta_{N-i+2}$ . From the ordering in the expansion (1.3) and the fact that

coefficients  $a_{\eta\nu}$  therein are positive, we see from the definition (3.55) that  $\binom{\nu}{\eta}$  is non-zero, and so can be cancelled from (4.18). Noting  $\eta_j^+ = \eta_{N-j+1}$ , we see from (3.63) that  $\frac{\epsilon_\nu - \epsilon_\eta}{2} = \eta_i^+ + \frac{N-i}{\alpha}$ , and so

$$\left(c + \eta_i^+ - \frac{i-1}{\alpha}\right) A_{\nu^+} = \alpha \left(a + \eta_i^+ - \frac{i-1}{\alpha}\right) \left(b + \eta_i^+ - \frac{i-1}{\alpha}\right) A_{\eta^+}.$$

This is a first order difference equation in the parts of the partition  $\eta^+$  and so, for a given initial condition ( $A_0 = 1$ ), has a unique solution. It is straightforward to verify from (4.7) that the solution is  $A_{\eta^+} = [a]_\eta [b]_\eta / [c]_\eta$ , as required.  $\square$

Of particular interest is Proposition 4.10 with the change of variables  $y \mapsto y/b$  and  $b \rightarrow \infty$ . This gives

**Corollary 4.11** *The p.d.e.*

$$\tilde{D}_1^{(x)} F + \left(c - \frac{N-1}{\alpha}\right) \tilde{E}_0^{(x)} F - \tilde{E}_2^{(y)} F = ap_1(y) F \quad (4.19)$$

is satisfied by

$$F = {}_1\mathcal{K}_1(a; c; x; y)$$

where

$${}_1\mathcal{K}_1(a; c; x; y) := \sum_{\eta} \alpha^{|\eta|} \frac{[a]_\eta}{[c]_\eta} \frac{d_\eta}{d'_\eta e_\eta} E_\eta(x) E_\eta(y).$$

By proceeding as in the derivation of [1, Prop. 4.2] we can deduce from Corollary 4.11 a generating function for the  $E_\eta^{(L)}$  involving the function  ${}_1\mathcal{K}_1$ .

**Proposition 4.12** *We have*

$$\prod_{i=1}^n (1 - z_i)^{-c-q} {}_1\mathcal{K}_1(c+q; a+q; -x; \frac{z}{1-z}) = \sum_{\eta} (-\alpha)^{|\eta|} \frac{[c+q]_\eta}{[a+q]_\eta} \frac{d_\eta}{d'_\eta e_\eta} E_\eta^{(L)}(x) E_\eta(z). \quad (4.20)$$

Replacing  $z$  by  $z/c$  in (4.20) and taking the limit  $c \rightarrow \infty$  we reclaim the generating function (4.12), while setting  $a = c$  the following generating function results.

**Proposition 4.13** *We have*

$$\prod_{i=1}^n (1 - z_i)^{-a-q} \mathcal{K}_A(-x; \frac{z}{1-z}) = \sum_{\eta} (-\alpha)^{|\eta|} \frac{d_\eta}{d'_\eta e_\eta} E_\eta^{(L)}(x) E_\eta(z), \quad (4.21)$$

where  $\mathcal{K}_A$  is defined by (3.17).

As in the case of the symmetric Laguerre polynomials, by using the generating function (4.21), the orthogonality of  $\{E_\eta^{(L)}\}$  with respect to (1.9) and Corollary 4.9 we can provide an alternative proof of Proposition 4.5 (see [1, Prop. 4.1] for details).

### 4.3 Generalized Hankel transform

The generating function formulas (4.12), (4.20) and (4.21) are all direct analogues of the generating functions for the symmetric Laguerre polynomials. In the symmetric case, the analogue of the function  $\mathcal{K}_B(x; y)$  also occurs as the kernel in an integral transform which relates the symmetric Laguerre and Jack polynomials [1, Prop. 4.11 and Cor. 4.1]. The non-symmetric analogue is easily obtained by following a similar strategy.

**Proposition 4.14** *Let*

$$d\mu^{(L)}(y) := \prod_{j=1}^n y_j^a e^{-y_j} \prod_{1 \leq j < k \leq n} |y_k - y_j|^{2/\alpha} dy_1 \dots dy_n. \quad (4.22)$$

We have

$$\int_{[0,\infty)^n} \mathcal{K}_B(x; -z_a) \mathcal{K}_B(x; -z_b) d\mu^{(L)}(x) = \mathcal{N}_0^{(L)} e^{-p_1(z_a)} e^{-p_1(z_b)} \mathcal{K}_B(z_a; z_b). \quad (4.23)$$

$$\int_{[0,\infty)^n} \mathcal{K}_B(x; -z_a) E_\eta^{(L)}(x) d\mu^{(L)}(x) = \mathcal{N}_0^{(L)} e^{-p_1(z_a)} E_\eta(-z_a). \quad (4.24)$$

$$\int_{[0,\infty)^n} \mathcal{K}_B(x; -z_a) E_\eta(-x) d\mu^{(L)}(x) = \mathcal{N}_0^{(L)} e^{-p_1(z_a)} E_\eta^{(L)}(z_a). \quad (4.25)$$

Using (4.25) it is straightforward to derive the Laguerre analogue of the summation formula in Proposition 3.14 (see [1, Prop. 4.12] for the symmetric case and further details).

**Proposition 4.15** *For  $|t| < 1$  we have*

$$\begin{aligned} \sum_\eta \frac{1}{\mathcal{N}_\eta^{(L)}} E_\eta^{(L)}(x) E_\eta^{(L)}(y) t^{|\eta|} &= \frac{1}{\mathcal{N}_0^{(L)}} (1-t)^{-N(a+q)} \\ &\times \exp\left(-\frac{t}{1-t} (p_1(x) + p_1(y))\right) \mathcal{K}_B\left(\frac{y}{1-t}; \frac{tx}{1-t}\right) \end{aligned}$$

We can use Proposition 4.15 to prove the analogue of the asymptotic expansion (3.33).

**Proposition 4.16** *We have*

$$\mathcal{K}_B\left(\frac{x^2}{2\tau}; \frac{y^2}{2\tau}\right) \sim \frac{\pi^{-n/2} 2^{n(a+1/2)+n(n-1)/\alpha} \mathcal{N}_0^{(L)}}{(\prod_{1 \leq j < k \leq n} (x_j^2 - x_k^2)(y_j^2 - y_k^2)/\tau)^{1/\alpha}} \prod_{j=1}^n (x_j y_j / \tau)^{-(a+1/2)} e^{x_j y_j / \tau}$$

as  $\tau \rightarrow 0$ .

*Proof.* This follows from the interpretation of the sum in Proposition 4.15 as the Green function for a Schrödinger equation. See the derivation of the symmetric counterpart of this result [1, eq. (5.47)] for further details (in fact, as is the case in the type A case (3.33), the asymptotic expansion is identical with its symmetric counterpart).

To further develop  $\mathcal{K}_B$  as an integral kernel we will require the exponential operator formula [2, eqs. (4.2)& (4.4)]

$$E_\eta^{(L)}(x^2) = e^{-\Delta_B/4} E_\eta(x^2), \quad (4.26)$$

which can also be derived from (4.11) and the generating function (4.12). Substituting (4.26) in (4.25) and (4.24) with the change of variables  $x \mapsto x^2$ ,  $z_a \mapsto z^2$ , and using the fact that  $\{E_\eta^{(L)}\}$  is a basis for analytic functions, we obtain the following formulas.

**Proposition 4.17** *We have*

$$\int_{(-\infty,\infty)^n} \mathcal{K}_B(x^2; -z^2) \left( e^{-\Delta_B/4} f(x^2) \right) d\mu^{(L)}(x^2) = \mathcal{N}_0^{(L)} e^{-p_2(z)} f(-z^2) \quad (4.27)$$

$$\int_{(-\infty,\infty)^n} \mathcal{K}_B(x^2; -z^2) f(-x^2) d\mu^{(L)}(x^2) = \mathcal{N}_0^{(L)} e^{-p_2(z)} e^{-\Delta_B/4} f(z^2) \quad (4.28)$$

where  $f$  is an analytic function such that all terms converge.

Analogous to the pairing (3.42) we define the  $B$ -type pairing

$$[p, q]_L = p((T^{(B)})^x) q(x), \quad (4.29)$$

where  $p$  and  $q$  are homogeneous polynomials of degree  $|\kappa|$  say. According to the theory of Dunkl [11, Th. 3.10], (4.29) is related to the exponential operator in (4.26) by

$$[p(x^2), q(x^2)]_L = \frac{2^{2|\kappa|}}{\mathcal{N}_0^{(L)}} \int_{(-\infty,\infty)^n} \left( e^{-\Delta_B/4} p(x^2) \right) \left( e^{-\Delta_B/4} q(x^2) \right) d\mu^{(L)}(x^2). \quad (4.30)$$

Using (4.26), the orthogonality of  $\{E_\eta^{(L)}\}$  with respect to (1.9) and Proposition 4.5 we obtain

**Proposition 4.18** *We have*

$$[E_\nu(x^2), E_\eta(x^2)]_L = 2^{2|\kappa|} \frac{[a+q]_\eta}{\alpha^{|\eta|}} \frac{d'_\eta e_\eta}{d_\eta} \delta_{\nu,\eta}. \quad (4.31)$$

Another derivation of (4.31) can be deduced from a recent work of Dunkl [7] on intertwining operators of type  $B$ . The key formula [7, proof of Th. 4.2] is that for  $f \in A_\lambda$  (this is the space defined above (3.47))

$$V^{(B)} \xi f(x^2) = \frac{1}{2^{2|\lambda|} [n/\alpha + 1]_\lambda^{(\alpha)} [a+q]_\lambda^{(\alpha)}} f(x^2) \quad (4.32)$$

with  $\xi$  defined below (3.47) and  $V^{(B)}$  defined by the intertwining relation  $T_i^{(B)} V^{(B)} = V^{(B)} \frac{\partial}{\partial x_i}$  and the normalization  $V 1 = 1$ . By following the working which led to (3.48) we deduce from (4.32) that for  $f, g \in A_\lambda$

$$[f(x^2), g(x^2)]_L = 2^{2|\lambda|} [n/\alpha + 1]_\lambda^{(\alpha)} [a+q]_\lambda^{(\alpha)} \sum_{\nu'} f_{\nu'} \sum_{\nu} A_{\nu'\nu} g_\nu. \quad (4.33)$$

Substituting (3.46) and using the facts noted below (3.49) reclaims (4.31).

The formula (4.27) has occurred in the type- $B$  theory of Dunkl [12, Prop. 2.1] with  $\mathcal{K}_B$  replaced by a certain kernel  $K_B(\sqrt{2}x, \sqrt{2}z)$ ,  $f(y)$  by  $f(\sqrt{2}y)$ ,  $x$  by  $x/\sqrt{2}$  and  $z$  by  $z/\sqrt{2}$ . Thus we must have

$$K_B(x, z) = \mathcal{K}_B(x^2/2; z^2/2). \quad (4.34)$$

We remark also that Dunkl [11] has proved, without using the explicit formula (4.34), that  $K_B$  has the properties (a) and (b) of Theorem 4.6. Furthermore, it follows from the properties of  $K_B$  established in [11] that  $\mathcal{K}_B$  is an entire function of all variables, and satisfies the uniform bound

$$|\mathcal{K}_B(x^2; z^2)| \leq e^{c p_1(|x|) p_1(|y|)}, \quad (4.35)$$

for some  $c > 0$  (c.f. Proposition 4.16).

#### 4.4 The generalized Hankel transform and its relation to the Laplace transform

In refs. [21, 34] (see also [1]) the generalized Hankel transform  $\mathcal{H}$  is defined as the symmetric version of

$$\mathcal{H}[f(x^2)](z^2) = \frac{1}{\mathcal{N}_0^{(L)}} \int_{(-\infty, \infty)^n} \mathcal{K}_B(x^2; -z^2) f(x^2) d\mu^{(L)}(x^2), \quad (4.36)$$

(i.e. (4.36) with  $\mathcal{K}_B(x^2; -z^2)$  replaced by  ${}_0\mathcal{F}_1(a+q; x^2; -z^2)$ ). In ref. [21] it is shown, on the basis of some conjectures concerning the generalized Laplace transform, that the symmetric Hankel transform is an isometry with respect to the inner product (1.8), and is further related to the Laplace transform by a generalization of Tricomi's theorem. In ref. [34] these properties are proved in the case  $n = 2$ .

In the non-symmetric case Dunkl [11, 12] has defined the Hankel transform by (4.36) with  $z^2$  replaced by  $iz^2$ . The isometry property is proved, and the relationship between the Hankel transform and polynomials annihilated by the operator  $\Delta_B$  explored in some detail. Here we complement Dunkl's theory by extending the results of ref. [21, 34] (see also [1]) for the symmetric case.

First we will calculate the generalized Laplace transform (3.67) of  $\prod_{j=1}^n x_j^a E_\eta^{(L)}$  and use the result to calculate the Laplace transform of  $\prod_{j=1}^n x_j^a E_\eta$ .

**Proposition 4.19** *We have*

$$\mathcal{L}\left(\prod_{j=1}^n x_j^a E_\eta^{(L)}(x)\right) = [a+q]_\eta \mathcal{N}_0^{(L)} \prod_{j=1}^n t_j^{-(a+q)} E_\eta(1 - \frac{1}{t}) \quad (4.37)$$

$$\mathcal{L}\left(\prod_{j=1}^n x_j^a E_\eta(x)\right) = [a+q]_\eta \mathcal{N}_0^{(L)} \prod_{j=1}^n t_j^{-(a+q)} E_\eta^{(L)}\left(\frac{1}{t}\right). \quad (4.38)$$

*Proof.* To derive (4.37) we multiply both sides of the generating function (4.21) by  $E_\eta^{(L)}(x)$  and integrate with respect to the measure  $d\mu^{(L)}(x)$  over the region  $[0, \infty)^n$ . This gives

$$\prod_{j=1}^n (1 - z_j)^{-(a+q)} \int_{[0, \infty)^n} \mathcal{K}_A(-x; \frac{z}{1-z}) E_\eta^{(L)}(x) d\mu^{(L)}(x) = \mathcal{N}_0^{(L)}[a+q]_\eta E_\eta(z) \quad (4.39)$$

where on the r.h.s. we have used the orthogonality of  $\{E_\eta^{(L)}\}$  with respect to (1.9) and the normalization in Proposition 4.5. Using the identity (3.53) we see that

$$\mathcal{K}_A(-x; \frac{z}{1-z}) d\mu^{(L)}(x) = \prod_{j=1}^n x_j^a \mathcal{K}_A(-x; \frac{1}{1-z}) dx_1 \dots dx_n.$$

Thus, with  $1/(1-z) = t$ , the integral on the left hand side of (4.39) is the generalized Laplace transform in (4.37), and the first result follows. The second result follows from the first by replacing  $\eta$  by  $\nu$ , and summing over  $\nu$  with appropriate  $\nu$  dependent factors given in (4.13) so that  $E_\eta^{(L)}(x)$  on the left hand side becomes  $E_\eta(x)$ . Performing the same operation on the r.h.s. we see from (3.55) and (4.13) that the resulting sum is equal to  $E_\eta^{(L)}(1/t)$ , as required.  $\square$

Using (4.37) we can compute the generalized Hankel transform of  $\mathcal{K}_A(-y^2; x^2)$ .

**Proposition 4.20** *We have*

$$\mathcal{H}[\mathcal{K}_A(-y^2; x^2)](z^2) = \mathcal{K}_A(-\frac{1}{y^2}; z^2) \prod_{j=1}^n y_j^{-2(a+q)}, \quad (4.40)$$

where  $y$  is regarded as a parameter.

*Proof.* From the definitions

$$\mathcal{H}[\mathcal{K}_A(-y^2; x^2)](z^2) = \frac{1}{\mathcal{N}_0^{(L)}} \mathcal{L}[\prod_{j=1}^n x_j^a \mathcal{K}_B(x; -z^2)](y^2).$$

Computation of the right hand side using (4.37) term-by-term on the series formula (4.10) gives the stated result.  $\square$

Note from (3.38) and (3.40) that we can use  $\mathcal{K}_A(-y^2; x^2)$  to form an arbitrary function of  $x^2$  for which the generalized Fourier transform theory is applicable. But from Proposition 4.20 we immediately have that

$$\mathcal{H}^2[\mathcal{K}_A(-y^2; x^2)](z^2) = \mathcal{K}_A(-y^2; z^2) \quad (4.41)$$

which when combined with the above remark says that in general  $\mathcal{H}$  is a projection operator:  $\mathcal{H}^2 = 1$ . We also have that  $\mathcal{H}$  is an isometry with respect to the inner product (1.9). To see this, note that

$$\begin{aligned} \langle \mathcal{K}_A(-y_1; x^2), \mathcal{K}_A(-y_2; x^2) \rangle_L &= \int_{(-\infty, \infty)^n} \mathcal{K}_A(-y_1; x^2) \mathcal{K}_A(-y_2; x^2) d\mu^{(L)}(x^2) \\ &= \prod_{j=1}^n (y_1)_j^{-(a+q)} {}_1\mathcal{K}_0(a+q; -\frac{1}{y_1}; y_2) \\ &= \prod_{j=1}^n (y_2)_j^{-(a+q)} {}_1\mathcal{K}_0(a+q; -\frac{1}{y_2}; y_1) \end{aligned} \quad (4.42)$$

where to obtain the second line we have used the series expansion (3.17) to replace  $\mathcal{K}_A(-y_2; x^2)$  and integrated term-by-term using (4.38), while to obtain the final line the symmetry with respect to the interchange of  $y_1$  and  $y_2$  has been used ( ${}_1\mathcal{K}_0$  is given by (4.15) with  $b = c$ ). On the other hand, from Proposition 4.20 we have

$$\langle \mathcal{H}[\mathcal{K}_A(-y_1; x^2)], \mathcal{H}[\mathcal{K}_A(-y_2; x^2)] \rangle_L = \prod_{j=1}^n (y_1)_j^{-(a+q)} (y_2)_j^{-(a+q)} \langle \mathcal{K}_A(-\frac{1}{y_1}; x^2), \mathcal{K}_A(-\frac{1}{y_2}; x^2) \rangle_L. \quad (4.43)$$

Substituting the second last expression of (4.42) with  $y_1, y_2$  replaced by  $1/y_1, 1/y_2$  in the right hand side we see that the final expression of (4.42) results, and so the right hand side of (4.43) is precisely  $\langle \mathcal{K}_A(-y_1; x^2), \mathcal{K}_A(-y_2; x^2) \rangle_L$ , as required.

Let us now relate the generalized Laplace transform and Hankel transforms, by giving the generalization of Tricomi's theorem.

**Proposition 4.21** *The formula*

$$g(z) = \mathcal{H}[f(x^2)](z) \quad (4.44)$$

*holds if and only if*

$$\mathcal{L}\left[\prod_{j=1}^n z_j^a g(z)\right](t) = \prod_{j=1}^n t_j^{-(a+q)} \mathcal{L}\left[\prod_{j=1}^n z_j^a f(z)\right]\left(\frac{1}{t}\right) \quad (4.45)$$

*Proof.* First assume (4.44). Since  $\mathcal{H}^2 = 1$  we have  $\mathcal{H}[g(x^2)](z) = f(z)$ . Use of this formula, the fact that  $\mathcal{H}$  is an isometry with respect to (1.9), and Proposition 4.20 gives

$$\begin{aligned} \mathcal{L}\left[\prod_{j=1}^n z_j^a g(z)\right](t) &= \langle \mathcal{K}_A(-t; z), g(z) \rangle_L \\ &= \langle \mathcal{H}[\mathcal{K}_A(-t; x^2)](z), \mathcal{H}[g(x^2)](z) \rangle_L \\ &= \prod_{j=1}^n t_j^{-(a+q)} \langle \mathcal{K}_A(-\frac{1}{t}; z), f(z) \rangle_L, \end{aligned}$$

which is equivalent to (4.45). Now assume (4.45). Proceeding as above we have

$$\begin{aligned} \prod_{j=1}^n t_j^{-(a+q)} \mathcal{L}\left[\prod_{j=1}^n z_j^a f(z)\right]\left(\frac{1}{t}\right) &= \langle \mathcal{H}[\mathcal{K}_A(-t; x^2)](z), f(z) \rangle_L \\ &= \langle \mathcal{K}_A(-t; z), \mathcal{H}[f(x^2)](z) \rangle_L \\ &= \mathcal{L}\left[\prod_{j=1}^n z_j^a \mathcal{H}[f(x^2)](z)\right] \end{aligned}$$

Comparing this equation with (4.45) and using the injectivity of the Laplace transform establishes (4.44).  $\square$

## 4.5 Relationship to Dunkl's theory of harmonic polynomials

In this subsection we will present the analogue of the results of subsection 3.5 in the type  $B$  case. The required theory to do this can be found in ref. [10].

The type  $B$  generalized harmonic polynomials of degree  $k$  in  $x_1^2, \dots, x_n^2$  are defined as the linearly independent solutions of the equation

$$\Delta_B \mathcal{Y}_{k,l}^B = 0.$$

Now, from [9, Th. 1.11], analogous to (3.70) we have

$$E_\eta(x^2) = \sum_{m=0}^{|\eta|} r^{2m} \mathcal{Y}_{|\eta|-m, \cdot}^B(x^2), \quad (4.46)$$

where  $r := (x_1^2 + \dots + x_n^2)^{1/2}$ , with

$$\mathcal{Y}_{|\eta|-m, \cdot}^B(x) = \frac{1}{4^m m! \binom{n(a+1) + n(n-1)/\alpha + 2|\eta| - 2m}{m}} \tilde{T}_{|\eta|-m}^B \left( \Delta_B^m E_\eta(x^2) \right), \quad (4.47)$$

$$\tilde{T}_k^B = \sum_{j=0}^k \frac{r^{2j}}{4^j j! \binom{-n(a+1) - n(n-1)/\alpha - 2k + 2}{j}} \Delta_B^j. \quad (4.48)$$

To use (4.46) to express the non-symmetric Laguerre polynomials in terms of the type  $B$  harmonic polynomials, we note from [11, Prop. 3.9] that

$$e^{-\Delta_B/4}|r|^{2j}\mathcal{Y}_{k,\cdot}^B(x) = (-1)^j j! L_j^{2k+n(n-1)/\alpha+n(a+1)-1}(r^2)\mathcal{Y}_{k,\cdot}^B(x), \quad (4.49)$$

Thus, applying (4.26) to (4.46) we obtain

$$E_\eta^{(L)}(x^2) = \sum_{m=0}^{|\eta|} (-1)^m m! L_m^{(|\eta|-m+n(n-1)/\alpha+n(a+1)-1)}(r^2)\mathcal{Y}_{|\eta|-m,\cdot}^B(x), \quad (4.50)$$

(cf. (3.74)).

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